NOTES FOR SEPTEMBER 21

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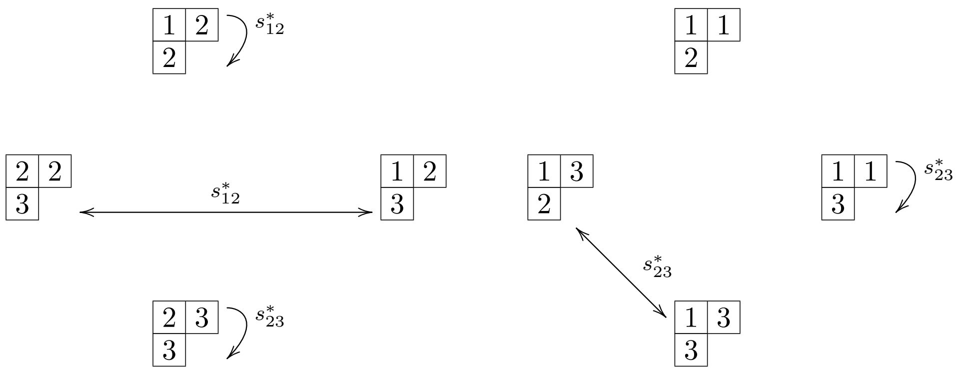


1. Conclusion of the proof of the Ratio of Alternants formula

Last time, we defined $\rho = (n-1, n-2, \cdots, 2, 1, 0)$. We have an action of S_n on $\mathbb{Z}^n$ by

$$v^*(\alpha) = v(\alpha) + v(\rho) - \rho$$

To finish the proof of the ratio of alternants formula, we must show the following.

Lemma 1.

$$\sum_{v \in S_n} (-1)^v K_{\lambda v^*(\alpha)} = \begin{cases} 1 & \lambda = \alpha \\ 0 & \textit{otherwise} \end{cases}$$

We will prove this by a sign-cancelling involution. Let T_{high} be the SSYT of shape λ with all 1's in its first row, all 2's in its second row, and so on.

$$T_{high} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \cline{1-4} 3 & 3 & \cdots \\ \cline{1-3} \vdots \\ \cline{1-1} n \\ \cline{1-1} \end{array}$$

Then T_{high} contributes to the $\alpha = \lambda$ term. Our involution will be defined on $SSYT(\lambda) \backslash \{T_{high}\}$, and will switch each tableau of content γ with one that has content $s_i^*(\gamma)$ for some s_i.

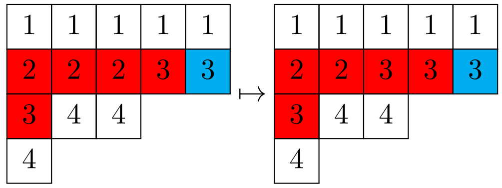
Recall that s_i is the permutation that switches i and $i+1$, so the action of s_i^* corresponds to keeping one $i+1$, and switching the rest of the i's and $i+1$'s. Thus

$$s_i^*(\gamma_1, \cdots, \gamma_i, \gamma_{i+1}, \cdots, \gamma_n) = (\gamma_1, \cdots, \gamma_{i+1} - 1, \gamma_i + 1, \cdots, \gamma_n)$$

For example, if (γ_i, γ_{i+1}) is $(2, 10)$, $s_i^*(\gamma)_{i,i+1}$ will be $(9, 3)$.

Now look at a tableaux T of shape Λ, and find the highest row that doesn't match T_{high}. Look at the last element of that row (shown in blue). This will be $i+1$. Let T contain r copies of i and s copies of $i+1$.

The boxes labeled i and $i+1$ in T form a smaller tableau within T (shown in red below), containing r copies of i and $s-1$ copies of $i+1$. We switch this inner tableau with one that has $s-1$ copies of i's and r copies of $i+1$ using the Bender-Knuth involution. So the new tableau as a whole contains $s-1$ copies of i and $r+1$ copies of $i+1$.



This process changes the content of T by s_i^*. There are no i's or $i+1$'s in the rows above the row where we differed from T_{high}. If the difference occurs in the k^{th} row, then $i+1 > k$. The elements in previous rows still match T_{high}. Elements in previous rows are $1, 2, \cdots, k-1$. So the highest rows still match T_{high}. The k^{th} row still doesn't match T_i because it still has the rightmost element, so the operation is self-inverting. The lemma is proved, and the the ratio of alternants formula follows.

2. Schur functions are orthonormal

We recall the Jacobi-Trudi formula

$$s_\mu = \begin{vmatrix} h_{\mu_1} & h_{\mu_1+1} & \cdots & h_{\mu_1+(n-1)} \\ h_{\mu_2-1} & h_{\mu_2} & \cdots & \\ \vdots & \vdots & \ddots & \\ h_{\mu_n-(n-1)} & & & h_{\mu_n} \end{vmatrix} = \sum_{w \in S_n} (-1)^w h_{w^*(\mu)}$$

(Note: for $\alpha \in \mathbb{Z}^n_{\geq 0}$, the notation h_α means $h_{\mathrm{sort}(\alpha)}$. For example, $s_{21} = \begin{vmatrix} h_2 & h_3 \\ h_0 & h_1 \end{vmatrix} = h_{21} - h_{03}$

We define

$$L_{\lambda\mu} = \sum_{w \in S_n} \begin{cases} (-1)^w & \text{if } w^*(\mu) \text{ is a permutation of } \lambda \\ 0 & \text{else} \end{cases}$$

so we have

$$s_\mu = \sum_{\lambda \text{ a partition}} L_{\lambda\mu} h_\lambda.$$

We also just proved

$$\sum_{w \in S_n} (-1)^w K_{\lambda w^*(\alpha)} = \begin{cases} 1 & \lambda = 0 \\ 0 & \text{else} \end{cases}$$

so

$$\sum_\nu K_{\lambda\nu} L_{\nu\alpha} = \delta_{\lambda\alpha}.$$

If two square matrices obey $AB = \mathrm{Id}$, they also[1] obey $BA = \mathrm{Id}$, so

$$\sum_\kappa L_{\lambda\kappa} K_{\kappa\mu} = \delta_{\lambda\mu}.$$

As a consequence

$$\sum_\kappa L_{\lambda\kappa} s_\kappa = m_\lambda.$$

Let $\langle\ ,\ \rangle$ be our standard inner product. Let $(\ ,\ )$ be the inner product where the Schur functions are orthonomal. Then

$$\langle m_\lambda, s_\mu \rangle = \left\langle m_\lambda, \sum_{\lambda'} L_{\lambda'\mu} h_{\lambda'} \right\rangle = L_{\lambda\mu} \quad \text{and} \quad (m_\lambda, s_\mu) = \left(\sum_\kappa L_{\lambda\kappa} s_\kappa, s_\mu \right) = L_{\lambda\mu}$$

By linearity, we have $\langle f, g \rangle = (f, g)$ for all $f, g \in \Lambda$. □

[1]It often happens in combinatorics that $AB = \mathrm{Id}$ has a simple combinatorial proof and $BA = \mathrm{Id}$ does not. Loehr and Mendes, *Bijective matrix algebra*, Linear Algebra Appl. **416** (2006), no. 2-3, 917–944 give an algorithm which, given as input a sign canceling involution proving $AB = \mathrm{Id}$, produces as output a sign canceling involution proving $BA = \mathrm{Id}$. Loehr and Mendes motivating example was to find a combinatorial interpretation of $\sum_\kappa L_{\lambda\kappa} K_{\kappa\mu} = \delta_{\lambda\mu}$. To my knowledge, no simpler interpretation is known. See also the discussion at `http://sbseminar.wordpress.com/2010/07/26/a-proof-length-challenge/`

3. Concluding remarks

This paragraph added by David. I can't remember whether I made this argument on Sept 21 or Sept 14, but it should be recorded somewhere. Let L be the part of Λ in degree n for some fixed n, so $L \cong \mathbb{Z}^N$ as an abelian group. Suppose that we have some other orthonormal basis t_λ for L. Then t_λ is some permutation of $\pm s_\lambda$. Proof: Write $t_\lambda = \sum c_{\lambda\mu} s_\mu$. Then $\langle t_\lambda, t_\lambda \rangle = \sum_\mu c_{\lambda\mu}^2 = 1$, indicating that precisely one of the $c_{\lambda\mu}$ is ± 1 and all others are zero. So $t_\lambda = \lambda s_{\lambda'}$ for some λ'. Moreover, if $\lambda \neq \mu$, then $\langle t_\lambda, t_\mu \rangle = 0$, so $\lambda' \neq \mu'$ and we see that $\lambda \mapsto \lambda'$ is a permutation.

Something David should have mentioned but didn't. Suppose that we have a sequence of homogenous symmetric polynomials t_i with integer coefficients, obeying $\prod (1 - x_i y_j)^{-1} = \sum t_i(x) t_i(y)$. Then the same proof shows t_i contains $\pm s_\lambda$ once for each λ, plus possibly some copies of the zero polynomial. We don't need to check first that t_i is a basis for Λ.

On problem set 1, problem 5, we show that ω preserves the inner product on Λ. So $\omega(s_\lambda) = \pm s_{\lambda'}$ for some λ'. On problem set 3, problem 3, we will show that $\omega(s_\lambda) = s_\lambda^T$. Sketch of proof: The Jacobi-Trudi identity gives s_λ as a determinant in the h's, so we can express $\omega(s_\lambda)$ as a determinant in the e's. Problem 3 evaluates that determinant and shows that it is s_{λ^T}.

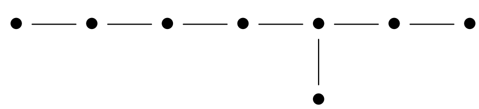
3.1. **A remark just for the fun of it.**

Suppose we have $L \cong \mathbb{Z}^N$, with inner product $L \times L \to \mathbb{Z}$ which is symmetric, positive definite, and has dual bases e_i and f_i both in L. We might expect that this would force a self-dual basis in L, but this is not the case.

For a counterexample, consider the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

corresponding to the graph



The matrix is symmetric, positive definite, and has determinant 1. But if $\langle \ , \ \rangle$ is the inner product corresponding to this matrix, then for any $v = (v_1, \ldots, v_8)$ we have

$$\langle v, v \rangle = 2 \sum v_i^2 - 2 \sum_{(i,j) \text{ in graph}} v_i v_j$$

and is hence even for all v (so in particular cannot be equal to 1).