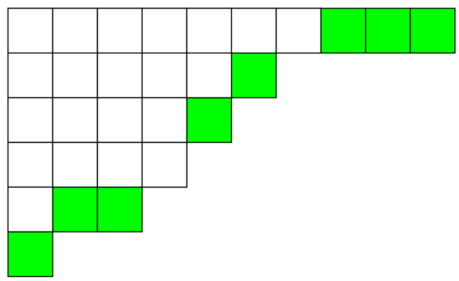
THE PIERI RULE – NOTES FOR SEPTEMBER 24

DAVID E SPEYER

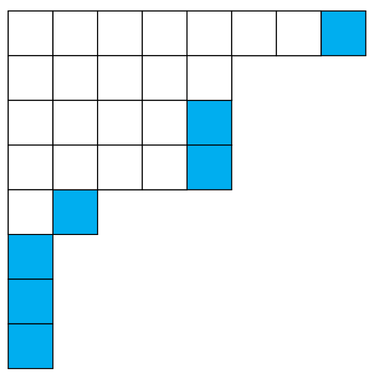
Today's task is to prove formulas for the products $h_k s_\lambda$ and $e_k s_\lambda$. These formulas are the Pieri and dual-Pieri rule:

$$h_k s_\lambda = \sum_{\nu/\lambda \text{ a horizontal strip}} s_\nu \quad \text{and} \quad e_k s_\lambda = \sum_{\nu/\lambda \text{ a vertical strip}} s_\nu$$

Of course, to make sense of these formulas, we need to define the terms "vertical strip" and "horizontal strip".



A vertical strip



A horizontal strip

In these figures, λ is the white inner partition (namely $(7, 5, 4, 4, 1)$), and ν is the set of all boxes, colored and white. Pictorially, in a horizontal strip, no two boxes of ν/λ are stacked on top of each other; in a vertical strip no two boxes of ν/λ are horizontally adjacent.

We can also write this in equations

$$h_k s_\lambda = \sum\nolimits_{\nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_n \geq \lambda_n \geq \nu_{n+1}} s_\nu. \quad (\dagger)$$

$$e_k s_\lambda = \sum\nolimits_{\substack{\nu_k = \lambda_k + (0 \text{ or } 1) \\ \nu_1 \geq \nu_2 \geq \cdots \geq \nu_r}} s_\nu. \quad (\dagger\dagger)$$

One formula is the image of the other under ω, so it suffices to prove one. We will prove the formula for $e_k s_\lambda$.

Fix n larger then $|\lambda| + k$; we will prove this formula for symmetric functions in n variables. We use our familiar notation $\Delta(\rho)$ for Vandermonde's determinants. So we want to show

$$e_k \Delta(\rho) s_\lambda = \sum_{\nu/\lambda \text{ a vertical strip}} \Delta(\rho) s_\nu$$

or

$$e_k \sum_{w \in S_n} (-1)^w x^{w(\lambda+\rho)} = \sum_{w \in S_n} (-1)^w \sum_{\nu/\lambda \text{ a vertical strip}} x^{w(\nu+\rho)}.$$

Expanding e_k, we want to show

$$\sum_{w \in S_n} \sum_{i_1 < i_2 < \cdots < i_k} (-1)^w x^{w(\lambda+\rho)+e_{i_1}+e_{i_2}+\cdots+e_{i_k}} = \sum_{w \in S_n} \sum_{\nu/\lambda \text{ a vertical strip}} (-1)^w x^{w(\nu+\rho)} \quad (*).$$

Here e_i is the i-th basis vector of $\mathbb{Z}^n$, the vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$.

We reuse an argument from the ratio of alternants lecture: We will compute the coefficient of x^β on both sides of the equation. Both sides of $(*)$ are antisymmetric, so it is enough to consider $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. If $\beta_i = \beta_{i+1}$, the coefficient is 0, so it is enough to consider $\beta_1 > \beta_2 > \cdots > \beta_n$.

Now, $\lambda_1 + \rho_1 > \lambda_2 + \rho_2 > \cdots > \lambda_n + \rho_n$. So the components of $w(\lambda + \rho)$ are ordered in the same way as w. In particular, they are decreasing if and only if $w = \mathrm{Id}$. Adding a $(0, 1)$ vector like $e_{i_1} + \cdots + e_{i_k}$ might make two unequal component become equal, but it won't reverse the order of two integers. So $w(\lambda + \rho) + e_{i_1} + e_{i_2} + \cdots + e_{i_k}$ has decreasing components only if $w = \mathrm{Id}$. So x^β appears on the left hand side if and only if β is of the form $\lambda + \rho + e_{i_1} + e_{i_2} + \cdots + e_{i_k}$ and $\beta_1 > \beta_2 > \cdots > \beta_n$. In other words, $\nu + \rho$ occurs if and only if $\nu = \lambda + e_{i_1} + \cdots + e_{i_k}$ and ν is a partition.

This is exactly the condition on the summation in (††).