NOTES FOR SEPTEMBER 26, 2012: SKEW SCHUR FUNCTIONS

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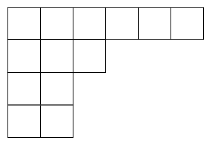
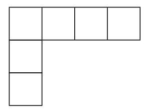
Let λ, μ be partitions such that $\lambda \supset \mu$, that is $\lambda_i \geq \mu_i$ for all i. The ***skew Young diagram*** λ/μ is the set difference between the two partitions.

Example 1. Let $\lambda = 6322$ and $\mu = 411$.

$\lambda =$

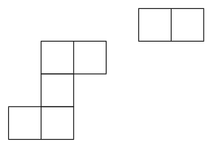


$\mu =$



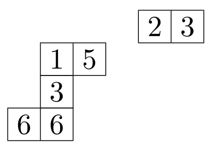
Then λ/μ is a skew Young diagram.

$\lambda/\mu =$



A ***(skew) semi-standard Young tableau*** (SSYT) of shape λ/μ is a filling of the Young diagram λ/μ with positive integers such that the rows are weakly increasing and the columns are strictly increasing.

Example 2. A semi-standard Young tableau of shape λ/μ.



The ***skew Schur function*** $s_{\lambda/\mu}$ is defined

$$s_{\lambda/\mu} = \sum_{\substack{\text{SSYT } T \\ \text{shape}(T)=\lambda/\mu}} x^T.$$

Proposition 1. *Skew Schur functions are symmetric.*

Proof. Same as that for Schur functions (via the Bender-Knuth involution). □

We now examine skew Schur functions with respect to various bases of Λ.

In the monomial basis, the skew Schur function is

$$s_{\lambda/\mu} = \sum_{\nu} K_{\lambda/\mu,\nu} m_\nu$$

where the coefficient $K_{\lambda/\mu,\nu}$ is the ***(skew) Kostka number***. It equals the number of SSYT of shape λ/μ and content ν.

Since ordinary Schur's already span Λ, the skew Schur's are linear combinations of these. The coefficients of this linear combination, as will be discussed below, are called Littlewood-Richardson numbers.

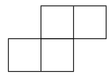
For the homogeneous basis, we have an analogue of the Jacobi-Trudi identity.

Proposition 2 (Jacobi-Trudi)**.**

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})$$

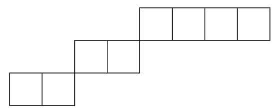
Proof. Same as that for Schur functions (non-intersecting lattice paths). □

Example 3.

$s_{32/1} =$  $\qquad s_{32/1} = \begin{vmatrix} h_2 & h_4 \\ h_0 & h_2 \end{vmatrix} = h_{22} - h_4$

Also, we have $\omega(s_{\lambda/\mu}) = s_{\lambda^T/\mu^T}$ and dual Jacobi-Trudi.

Any product of skew Schur functions is also a skew Schur function, as we can put two skew shapes together (disconnected-ly) to make a new skew shape for the product. In particular, any complete homogeneous polynomial h_λ is a skew Schur function. This is because $h_k = s_{(k)}$, where (k) is a row of k boxes. Then (for example) $h_{422} = s_{(4)}s_{(2)}s_{(2)}$ is the skew Schur function for the shape



which has disconnected rows of length $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$.

There is also a relation between skew Schur and non-skew Schur functions.

Proposition 3. *The "skew by μ" operator that sends $s_\lambda \mapsto s_{\lambda/\mu}$ is adjoint to multiplication by s_μ. In other words, for any $f \in \Lambda$*

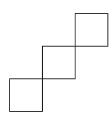
$$\langle s_{\lambda/\mu}, f \rangle = \langle s_\lambda, f s_\mu \rangle .$$

In particular, for a Schur function s_ν

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle .$$

Then the decomposition of $s_{\lambda/\mu}$ into a sum of non-skew Schur functions s_ν has structure constants that are the constants that you get from multiplying non-skew Schur functions.

Example 4. For the skew shape 321/21



we can compute $s_{321/21}$. Because $s_{321/21}$ is the product of the skew Schurs of its disconnected components

$$s_{321/21} = (s_1)^3 = (s_2 + s_{11})s_1 = (s_3 + s_{21}) + (s_{21} + s_{111}) = s_3 + 2s_{21} + s_{111}.$$

We can check that the coefficient of s_3 in $s_{321/21}$ (expanded in the s-basis) is

$$\langle s_{321/21}, s_3 \rangle = \langle s_{321}, s_3 s_{21} \rangle = 1$$

because

$$s_3 s_{21} = s_{321} + s_{411} + s_{42} + s_{51}$$

which has s_{321} with coefficient 1.

Recall the Pieri rule, which we use for some computations in the above. A horizontal k-strip is a skew shape with k boxes and no more than one box in each column. The Pieri rule is

$$s_\mu h_k = \sum_\lambda s_\lambda$$

where the sum ranges over all partitions λ such that λ/μ is a horizontal k-strip.

Proof. (of Proposition 3, adjointness) It suffices to prove this for the f from some basis for Λ, so we will show it for $f = h_\nu$. We wish to show that $\langle s_{\lambda/\mu}, h_\nu \rangle = \langle s_\lambda, s_\mu h_\nu \rangle$.

On the LHS, we have

$$\begin{aligned} \langle s_{\lambda/\mu}, h_\nu \rangle &= \text{coefficient of } m_\nu \text{ if } s_{\lambda/\mu} \text{ is written in the } m\text{-basis} \\ &= K_{\lambda/\mu,\nu} \qquad \text{(that is, the number of SSYT of shape } \lambda/\mu\text{, content } \nu\text{).} \end{aligned}$$

On the RHS

$$\langle s_\lambda, s_\mu h_\nu \rangle = \text{coefficient of } s_\lambda \text{ in } s_\mu h_\nu.$$

Use the Pieri rule to turn $s_\mu h_\nu$ into a sum.

$$\begin{aligned}
s_\mu h_\nu &= s_\mu h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} \\
&= \sum_{\rho^{(1)}} s_{\rho^{(1)}} h_{\nu_2} h_{\nu_3} \cdots h_{\nu_k} && \begin{cases} \rho^{(1)} \text{ s.t. } \rho^{(1)}/\mu \text{ is a horizontal } \nu_1\text{-strip} \end{cases} \\
&= \sum_{\rho^{(1)},\rho^{(2)}} s_{\rho^{(2)}} h_{\nu_3} \cdots h_{\nu_k} && \begin{cases} \rho^{(1)} \text{ s.t. } \rho^{(1)}/\mu \text{ is a horizontal } \nu_1\text{-strip} \\ \rho^{(2)} \text{ s.t. } \rho^{(2)}/\rho^{(1)} \text{ is a horizontal } \nu_2\text{-strip} \end{cases} \\
&= \sum_{\rho^{(1)},\rho^{(2)},\ldots,\rho^{(k)}} s_{\rho^{(k)}} && \begin{cases} \rho^{(1)} \text{ s.t. } \rho^{(1)}/\mu \text{ is a horizontal } \nu_1\text{-strip} \\ \vdots \\ \rho^{(k)} \text{ s.t. } \rho^{(k)}/\rho^{(k-1)} \text{ is a horizontal } \nu_k\text{-strip} \end{cases}
\end{aligned}$$

Each term $s_{\rho^{(k)}}$ in the sum represents a SSYT with shape $\rho^{(k)}/\mu$ and content ν; in the SSYT, each cell of the ith horizontal strip contains i. That is,

$$s_\mu h_\nu = \sum_{\substack{\text{SSYT } T \\ \text{shape}(T)=\rho/\mu \\ \text{content}(T)=\nu}} s_\rho.$$

The coefficient of s_λ in $s_\mu h_\nu$ is the number of SSYT with shape λ/μ and content ν, i.e. $K_{\lambda/\mu,\nu}$, and

$$\langle s_{\lambda/\mu}, h_\nu \rangle = \langle s_\lambda, s_\mu h_\nu \rangle$$

as desired. □

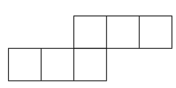
Example 5. If $\mu = (k)$, then

$$\langle s_{\lambda/(k)}, s_\nu \rangle = \langle s_\lambda, s_\nu s_{(k)} \rangle = \begin{cases} 1 & \text{if } \lambda/\nu \text{ is a horizontal } k\text{-strip} \\ 0 & \text{otherwise} \end{cases}$$

and

$$s_{\lambda/(k)} = \sum_{\lambda/\nu \text{ is a horizontal } k\text{-strip}} s_\nu.$$

With some actual numbers,

$53/2 =$  $\qquad s_{53/2} = s_{33} + s_{42} + s_{51}.$

The coefficients $\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle = c^\lambda_{\mu\nu}$ are called ***Littlewood-Richardson coefficients***. One can also define $s_{\lambda/\mu}$ with the following relation.

$$s_\lambda(x, y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda/\mu}(y)$$

This is basically the same as the definition by summing over skew SSYT. Take a SSYT of shape λ and consider the positions of the entries indexing x variables. These form some SSYT of some shape μ; the remaining y-variables form an SSYT of shape λ/μ.

We reprove the adjontness relation from this perspective.

Proof.

$$\begin{aligned}\prod_{i,j}(1-x_iz_j)^{-1}\times\prod_{i,j}(1-y_iz_j)^{-1} &= \sum_{\lambda} s_\lambda(x,y)s_\lambda(z)\\ &= \sum_{\lambda,\mu} s_\mu(x)s_{\lambda/\mu}(y)s_\lambda(z)\\ &= \sum_{\lambda,\mu,\nu} s_\mu(x)s_\nu(y)s_\lambda(z)\left\langle s_{\lambda/\mu}, s_\nu\right\rangle\end{aligned}$$

and

$$\begin{aligned}\prod_{i,j}(1-x_iz_j)^{-1}\times\prod_{i,j}(1-y_iz_j)^{-1} &= \left(\sum_{\mu} s_\mu(x)s_\mu(z)\right)\left(\sum_{\nu} s_\nu(y)s_\nu(z)\right)\\ &= \sum_{\mu,\nu} s_\mu(x)s_\nu(y)s_\mu(z)s_\nu(z)\\ &= \sum_{\lambda,\mu,\nu} s_\mu(x)s_\nu(y)s_\lambda(z)\langle s_\lambda, s_\mu s_\nu\rangle\end{aligned}$$

so $\left\langle s_{\lambda/\mu}, s_\nu\right\rangle = \langle s_\lambda, s_\mu s_\nu\rangle$. □