NOTES FOR SEPTEMBER 28, 2012: COMPACT GROUPS AND HAAR MEASURE

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Recall that a representation of a group G is a map of groups $G \to \mathrm{GL}(V)$ for some vector space V. For us, V will always be a complex vector space.

Our route to studying the representations of $\mathrm{GL}_n(\mathbb{C})$ is through the unitary group $\mathrm{U}(n)$. Recall that $\mathrm{U}(n)$ is the group of matrices that preserve the norm $\sum_{i=1}^n |z_i|^2$. In terms of matrices, a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ is unitary if $U\overline{U}^{\mathrm{t}} = \mathrm{Id}$. An important feature is that $\mathrm{U}(n)$ is a compact group. We now focus our work on compact groups to start our journey.

1. Compact Groups

Definition 1.1. A topological group G is a group and a topological space such that multiplication $G \times G \to G$ and inversion $G \to G, g \mapsto g^{-1}$ are continuous.

Remark. In our world, all topological spaces are Hausdorff.

Definition 1.2. A continuous representation of a topological group G means that the map $\rho : G \to \mathrm{GL}(V)$ is continuous.

2. The Benefits of Haar Measure

The representation theory of compact groups is much nicer than the representation theory of non-compact groups. The reason for this is that in the context of compact groups, we have a notion of "averaging" that we do not have in the non-compact group setting. We discuss this precisely now.

Let G be a compact group and let $f : G \to \mathbb{C}$ be a complex-valued function on G. If G is finite, we can compute the average $\frac{1}{|G|}\sum_{g \in G} f(g)$. We would like to generalize this to a compact group G.

There is something called *Haar measure* which lets us talk about $\int_G f(g)\,dg$ for measurable functions f. (In particular, we may do this for when f is continuous.) The key properties of the Haar measure are the following:

- It is linear.
- $\int_G 1\,dg = 1$.
- It is left- and right-invariant. That is, for any $h \in G$,
$$\int_G f(hg)\,dg = \int_G f(gh)\,dg = \int_G f(g)\,dg.$$
- If $f : G \to \mathbb{R}_{\geq 0}$, then
$$\int_G f(g)\,dg \geq 0.$$
Furthermore, if f is continuous, $f \geq 0$ and $\int_G f(g)\,dg = 0$, then f must be identically 0.

Remark. On locally compact topological groups, we always have a left Haar measure. The problem is that the right Haar measure might not align with the left Haar measure. If these two do coincide, then we say that the group is *unimodular*.

Proposition 2.1. *Let G be a compact group an $\rho : G \to \mathrm{GL}(V)$ a representation. Then G preserves a positive definite Hermitian form on V.*

Proof. Choose an arbitrary basis for V and define an inner product
$$\langle u, v\rangle = \int_G \rho(g)u \cdot \overline{\rho(g)v}\,dg.$$

This is clearly Hermitian and symmetric. It is positive definite as

$$\langle v, v\rangle = \int_G |\rho(g)v|^2 \, dg \geq 0.$$

The compact group G preserves this form since for any $h \in G$,

$$\langle hu, hv\rangle = \int_G \rho(h)\rho(g)u \cdot \overline{\rho(h)\rho(g)v} \, dg = \int_G \rho(g')v \cdot \overline{\rho(g')v} \, dg' = \langle u, v\rangle,$$

where the main step in the above computation was the substitution $g' = hg$. □

Since all positive-definite Hermitian forms are equivalent, we can choose a basis where $\langle e_i, e_j\rangle = \delta_{ij}$. Then this means that the image of G under ρ lies in the unitary group.

Lemma 2.1. *Let G be a compact group and let $\rho : G \to \mathrm{GL}(V)$ be a continuous representation. Let $U \subseteq V$ be a subrepresentation of V. Then there exists another subrepresentation W such that $V = U \oplus W$.*

Proof. Find a G-invariant positive-definite Hermitian form $\langle \cdot, \cdot\rangle$. Then take W to be the orthogonal complement of U with respect to this form. □

Definition 2.1. A representation W of G is called *simple* if it has no subrepresentations other than 0 and itself.

Corollary 2.1. *If G is compact, then every continuous representation of G is a direct sum of simples representations.*

Remark. A counterexample to the corollary when G is non-compact is if we take $G = \mathbb{Z}$ and consider the representation $\mathbb{Z} \to \mathrm{GL}_2(\mathbb{F})$ given by $k \mapsto \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$.

This is all fine and dandy. But let's PAUSE FOR SOME QUESTIONS: Why is there Haar measure? Why is it both left- and right-invariant?

3. Obtaining a Haar Measure for Compact Groups

Let G be a Lie group of dimension n. Choose a volume form ω on the tangent space at the identity—call it T_eG. Build an n-form η on G by translating ω according to the left action. Set

$$\eta|_{T_gG} = L_{g^{-1}}^*\omega.$$

Integration against η gives a notion of $\int_G$ with

$$\int_G f(g) \, dg = \int_G f(hg) \, dg.$$

Moreover, every left-invariant integration is of this form for some ω.

Note that $R_h^*\eta$ also gives a left-integration and

$$(R_h^*\eta)|_{T_{h'}G} = R_h^*(\eta|_{T_{h'h^{-1}}G}) = R_h^* L_{h(h')^{-1}}^*\omega = L_{h(h')^{-1}}^* R_h^*\omega = L_{(h')^{-1}}^*(\text{stuff}).$$

So $R_h^*\eta$ is a left Haar measure, but it may differ from the original left Haar measure η.

Remark: This is a problem for groups in general. For example, consider $\mathbb{R}_{>0} \ltimes \mathbb{R}$, realized by $\left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$ for $u > 0$. Left Haar measure is $\eta = (du \wedge dv)/u^2$. Pulling back $(du \wedge dv)/u^2$ by the right action of $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ gives $(d(au) \wedge d(bu + v))/(au)^2 = (du \wedge dv)/(au^2) = \eta/a$.

Since every left invariant Haar measure is of the form $\alpha\eta$, we have

$$R_h^*\eta = \alpha(h)\eta$$

for some $\alpha \colon G \to \mathbb{R}$ and $\alpha(h_1h_2) = \alpha(h_1)\alpha(h_2)$. Now, G is compact, and hence α is bounded, which forces $\alpha(G) \subseteq \{\pm 1\}$. It turns out that $\alpha(G) = \{1\}$.