NOTES FOR SEPTEMBER 5

DAVID E SPEYER

We will be studying the representation theory of $GL_n(\mathbb{C})$. This is a big topic, and we will only study a little of it. In particular, we will be looking at the connections to combinatorics. Classically, this means symmetric polynomals and tableaux. We'll also try to talk about some more modern perspectives, like crystals, Gelfand-Tsetlin patterns and webs, though probably not all of these.

1. The kind of representations we care about

For any group G, a representation is a vector space V and a map of groups $\rho : G \to GL(V)$. We are interested in the case where G is also $GL_n(\mathbb{C})$, so we are looking at maps $GL_n(\mathbb{C}) \to GL_N(\mathbb{C})$.

Write U for the standard n-dimensional representation of $GL_n(\mathbb{C})$.

Some examples of representations we want to study:

(1) We can take $V = U$.
(2) We can take $V = \bigwedge^k U$ for some k.
(3) We can take $V = \mathrm{Sym}^k U$ for some k.
(4) We can take $V = U^\vee$. In coordinates, this is $g \mapsto g^{-T}$.
(5) We can take $V = \mathbb{C}$ and $\rho(g) = (\det g)^k$ for some integer k.
(6) We can take $V = \left(\mathrm{Sym}^2 U \otimes U\right) \cap \left(U \otimes \bigwedge^2 U\right)$, where the intersection is inside $U \otimes U \otimes U$. This is an example of a Schur functor.

To understand example (6), recall that, if $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are representations, then $g \mapsto \rho(g) \otimes \sigma(g)$ gives an action of G on $V \otimes W$.

Some representations we don't want to study: The field $\mathbb{C}$ has tons of automorphisms, assuming the axiom of choice. We could compose any of the above examples with one of these automorphisms, and get some bizarre representation. We don't even want to consider the case where the automorphism is complex conjugation; that is, we don't want to consider $\rho(g) = \overline{g}$. Example (5) is an example we want. But $\rho(g) = |\det g|$ is not something we want to consider, and we certainly don't want to consider raising $|\det g|$ to non-integer powers.

A representation $\rho : GL_n(\mathbb{C}) \to GL_N(\mathbb{C})$ is a ***polynomial representation*** if the entries of the matrix $\rho(g)$ are polynomials in the entries of g. Of the above examples, (1), (2), (3) and (6) are polynomial. (5) is polynomial if $k \geq 0$, but not for $k < 0$. (4) is not polynomial, because g^{-1} is computed by taking the adjugate matrix and diving by $\det g$, so there will be powers of $\det g$ in the denominator. Motivated by these examples, we come up with the slightly more general definition: A representation ρ is a ***rational representation*** if the entries of $\rho(g)$ are rational functions of the entries of g, whose denominators are powers of $\det g$. We will be studying polynomial and rational representations.

Why rational representations? From the perspective of algebraic geometry, these are the representations where the map $\rho : GL_n(\mathbb{C}) \to GL_N(\mathbb{C})$ is an algebraic map. Recall that the coordinate ring of $GL_n(\mathbb{C})$ is generated by the matrix entries *and* $(\det g)^{-1}$. So ρ is a rational map if it is given by functions in the coordinate ring.

Another reason is that, if we instead ask the map ρ to be given by holomorphic functions, we get the exact same set of representations. So we wind up studying the same set of representations with fewer analytic prerequisites by looking at rational representations.

A third reason is that, if we want to think about representations of $GL_n(k)$ for some other field k, polynomiality and rationality are the only reasonable niceness conditions we have available to us.

2. The relation between representations and symmetric polynomials

If $\rho : G \to GL(V)$ is an finite dimensional representation, then the character χ of ρ is defined by $\chi(g) = Tr(\rho(g))$. Note that

$$\chi(hgh^{-1}) = Tr(\rho(hgh^{-1})) = Tr(\rho(h)\rho(g)\rho(h)^{-1}) = Tr(\rho(h)^{-1}\rho(h)\rho(g)) = Tr(\rho(g)) = \chi(g)$$

so χ is constant on conjugacy classes. Now, the diagonalizable matrices are dense in $GL_n(\mathbb{C})$, so any continuous function on $GL_n(\mathbb{C})$ is determined by its values on diagonalizable matrices. And "diagonalizable" just means "conjugate to diagonal". So characters of continuous representations are determined by their values on diagonal matrices.

If ρ is polynomial, then $\chi\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ is a polynomial in $x_1, x_2, \ldots, x_n$. If ρ is rational, then it is a Laurent polynomial.

Moreover, note that

$$\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}^{-1} = \begin{pmatrix} y & & \\ & z & \\ & & x \end{pmatrix}$$

and, more generally, conjugating a diagonal matrix by a permutation matrix permutes its entries. So $\chi\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ is a *symmetric* (Laurent) polynomial in $x_1, x_2, \ldots, x_n$. In this way, we get a symmetric (Laurent) polynomial from each polynomial (rational) respresentation.

We introduce the notation Λ_n for the ring of symmetric polynomials in $x_1, x_2, \ldots, x_n$, with integer coefficients. We'll write $\Lambda_n^{\pm}$ for symmetric Laurent polynomials. Soon we will also introduce Λ and $\Lambda^{\pm}$, which conceptually are the limits of Λ_n and $\Lambda_n^{\pm}$ as $n \to \infty$.

3. The main relations between representations and symmetric polynomials

The following are our main results for the first month and a half.

- A (polynomial or rational) representation is determined up to isomorphism by its character.
- A representation of GL_n is a direct sum of simple representations; here "simple" means "having no nontrivial subreps". There is a classical basis s_λ of Λ called the ***Schur polynomials***, corresponding to the simple reps.
- There is an inner product $\langle\ ,\ \rangle$ on Λ, called the Hall inner product, such that
$$\langle \chi_V, \chi_V \rangle = \dim \mathrm{Hom}_{GL_n(\mathbb{C})}(V, W).$$
- Restriction from $GL_n(\mathbb{C})$ to $U(n)$ gives an isomorphism of categories from {Rational $GL_n(\mathbb{C})$ reps} to {Smooth (or continuous, or real analytic, or measurable ...) $U(n)$ reps}.

4. Course outline

In rough outline, the plan is the following:

Sept	Lightning introduction to symmetric polynomials.
Oct	Connections between symmetric polynomials and representation theory.
Early Nov	Classical tableaux theory: RSK, jdt, Littlewood-Richardson, etc.
Late Nov-Dec	More modern tools: Some subset of crystals, webs, Gelfand-Tsetlin bases, honeycombs.

See the course website: `http://www.math.lsa.umich.edu/~speyer/665.html` for course policies and beauracracy.

5. Basics of partitions and symmetric polynomials

For $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, we define the element $m_\lambda \in \Lambda_n$ as demonstrated by the following examples:

$$m_{321}(x, y, z) = x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3 \text{ in } \Lambda_3.$$

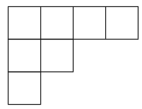
$$m_{1100}(w, x, y, z) = wx + wy + wz + xy + xz + yz \text{ in } \Lambda_4.$$

Note that we do NOT write wx twice, even though there are two ways to write the variables (w, x, y, z) under the exponents $(1, 1, 0, 0)$ to get the monomial wx. We frequently omit zeroes, so $m_{11} = m_{1100}$.

The m_λ form a basis for Λ_n, as λ ranges through n-tuples $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. To get a basis for $\Lambda^{\pm}$, just omit the condition that the λ_i be ≥ 0.

A ***partition*** is a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We treat trailing zeroes as negligible, so $(4, 2, 1)$, $(4, 2, 1, 0)$, $(4, 2, 1, 0, 0)$, ... are the same partition.

We often represent a partition by a ***young diagram***. For example

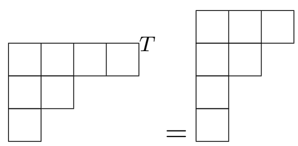


is $(4, 2, 1)$.

We define ***majorization order*** by $\lambda \preceq \mu$ if

$$\begin{aligned} \lambda_1 &\leq \mu_1 \\ \lambda_1 + \lambda_2 &\leq \mu_1 + \mu_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &\leq \mu_1 + \mu_2 + \mu_3 \\ &\text{and so forth.} \end{aligned}$$

We define ***transpose*** of partitions by example:



so $(4, 2, 1)^T = (3, 2, 1, 1)$.