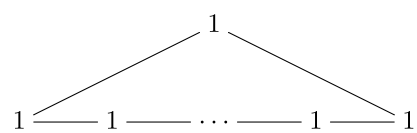
CLASSIFICATION OF POSITIVE-DEFINITE COXETER GRAPHS

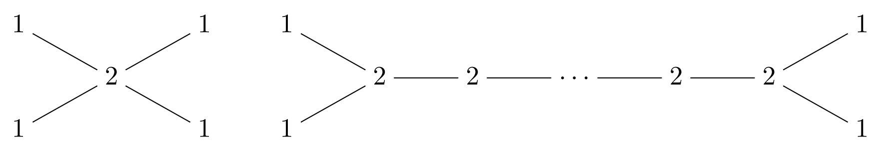
In these diagrams, unlabeled edges have $m_{ij} \geq 3$ (so $A_{ij} = -2\cos\frac{\pi}{m_{ij}} \leq -1$). Edge labels show lower bounds for m_{ij}. We remind the reader of a few key values:

m_{ij}	A_{ij}
2	0
3	-1
4	$-\sqrt{2}$
5	$-\frac{1+\sqrt{5}}{2}$
6	$-\sqrt{3}$

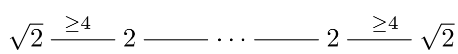
For each graph, I give a vector with nonpositive self inner product.



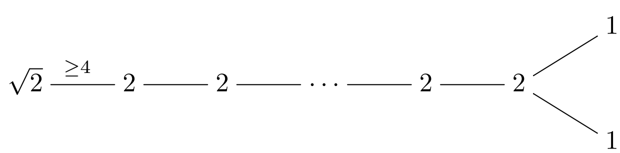
This implies Γ is a tree.



This implies Γ has no vertices of degree ≥ 4, and at most one of degree 3.



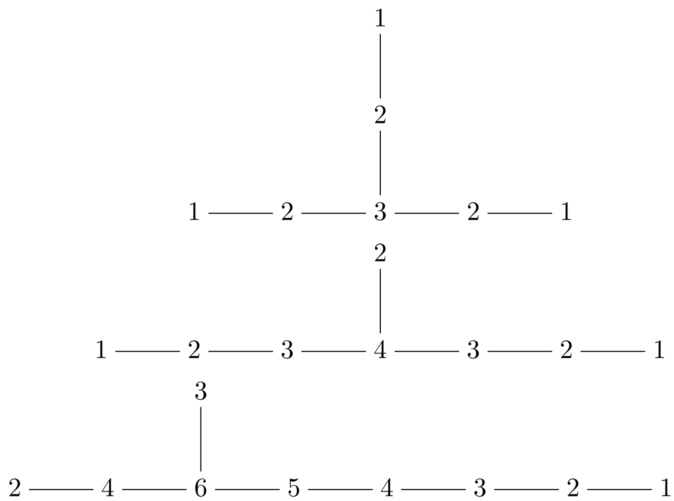
This implies Γ has at most one edge with $m_{ij} \geq 4$.



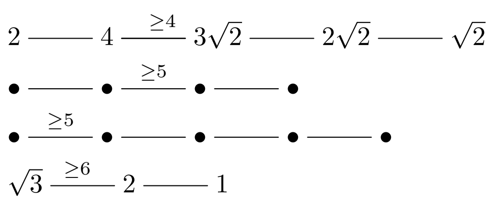
This implies Γ does not have both an edge with $m_{ij} \geq 4$ and a vertex of degree 3. At this point, we know that Γ is either

(1) 3 paths with a common endpoint, and all edges with $m_{ij} = 3$ or
(2) A single path, with at most one edge having $m_{ij} > 3$.

The following excluded graphs rule out all but finitely many three path cases:



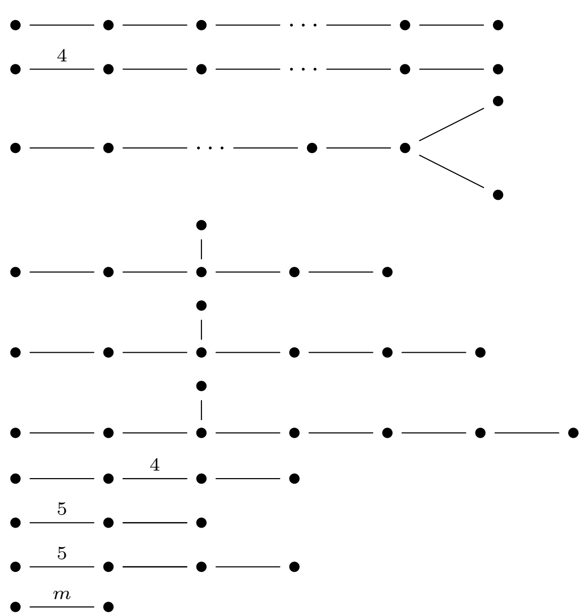
The following excluded graphs rule out all but finitely many single path cases. In the middle two cases, there is no particularly natural choice of vector with negative inner product, so we just compute the determinant.



$\det = 3 - 2\sqrt{5} < 0$

$\det = 4 - 2\sqrt{5} < 0$

The surviving graphs are the (connected) Coxeter diagrams! Note that edge labels now mean exact values, not lower bounds. Unlabeled edges now mean $m_{ij} = 3$.



A_n

B_n

D_n

E_6

E_7

E_8

F_4

H_3

H_4

$I_2(m)$

There are also some alternate names:

$$B_n = C_n \quad A_2 = I_2(3) \quad B_2 = I_2(4) \quad G_2 = I_2(6) \quad A_3 = D_3.$$