# COXETER GROUPS

Class notes from Math 665, taught at the University of Michigan, Fall 2017. Class taught by David E Speyer; class notes edited by David and written by the students: Elizabeth Collins-Wildman, Deshin Finlay, Haoyang Guo, Alana Huszar, Gracie Ingermanson, Zhi Jiang and Robert Walker. Comments and corrections are welcome!

## Contents

**September 6 – Introduction.** This course will study the combinatorics and geometry of Coxeter groups. We are going to spend the first few days discussing the finite reflection groups. We are doing this for three reasons:

(1) The classification of finite reflection groups is the same as that of finite Coxeter groups, and the structure of finite reflection groups motivates the definition of a Coxeter group.
(2) One of my favorite things in math is when someone describes a natural sort of object to study and turns out to be able to give a complete classification. Finite reflection groups is one of those success stories.
(3) This will allow us to preview many of the most important examples of Coxeter groups. I'll also take the opportunity to tell you their standard names.

Today we will, without proof, describe all the finite reflection groups.

Let $V$ be a vector space with a positive definite symmetric bilinear form $\cdot$ (dot product). Let $\alpha$ be a nonzero vector in $V$. The ***orthogonal reflection*** across $\alpha^\perp$ is the linear map

$$x \mapsto x - 2\frac{\alpha \cdot x}{\alpha \cdot \alpha}\alpha.$$

This fixes the hyperplane $\alpha^\perp$ and negates the normal line $\mathbb{R}\alpha$. (The word "orthogonal" is there in anticipation of future weeks, when we'll have non-orthogonal reflections.) A ***orthogonal reflection group*** is a subgroup of $\mathrm{GL}(V)$ generated by reflections. We will be classifying the finite orthogonal reflection groups.

Let $W_1 \subset \mathrm{GL}(V_1)$ and $W_2 \subset \mathrm{GL}(V_2)$ be reflection groups. Then we can embed $W_1 \times W_2$ into $\mathrm{GL}(V_1 \oplus V_2)$, sending $(w_1, w_2)$ to $\left(\begin{smallmatrix} w_1 & 0 \\ 0 & w_2 \end{smallmatrix}\right)$. We call a reflection group ***irreducible*** if we can not write it as a product in this way. We will now begin describing all of the irreducible orthogonal reflection groups.

**The trivial group:** Take $V = \mathbb{R}$ and $W = \{1\}$.

**The group of order 2:** Take $V = \mathbb{R}$ and $W = \{\pm 1\}$. This group can be called $A_1$ or $B_1$.

**The dihedral group:** Let be a positive integer and consider the group of symmetries of regular $m$-gon in $\mathbb{R}^2$. This is generated by reflections across two hyperplanes with angle $\pi/m$ between them. Calling these reflections $\sigma$ and $\tau$, we have $\sigma^2 = \tau^2 = (\sigma\tau)^m = 1$. This dihedral group has order $2m$ and can be called $I_2(m)$. We note a general rule of naming conventions – the subscript is always the dimension of $V$.

**The symmetric group:** Consider the group $S_n$ acting on $\mathbb{R}^n$ by permutation matrices. The transposition $(ij)$ is the reflection across $(e_i - e_j)^\perp$. This breaks down as $\mathbb{R}(1,1,\ldots,1) \oplus (1,1,\ldots,1)^\perp$ So the irreducible reflection group is $S_n$ acting on $(1,1,\ldots,1)^\perp$. This is called $A_{n-1}$.

**The group $S_n \ltimes \{\pm 1\}^n$:** Consider the subgroup of $GL_n(\mathbb{R})$ consisting of matrices which are like permutation matrices, but with a $\pm 1$ in the nonzero positions. This is a reflection group, generated by the reflections over $(e_i - e_j)^\perp$, $(e_i + e_j)^\perp$ and $e_i^\perp$. We give example matrices for $n = 3$ below:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reflection across $(e_1 - e_2)^\perp$ $(e_1 + e_2)^\perp$ $e_1^\perp$

For reasons we will get to eventually, this is both called $B_n$ and $C_n$.

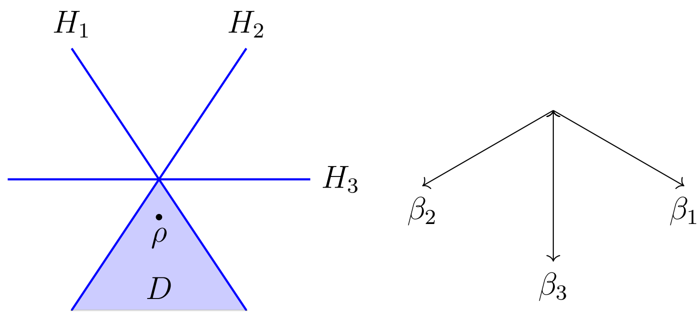



FIGURE 1. Our running example of a hyperplane arrangement

**The group $S_n \ltimes \{\pm 1\}^{n-1}$:** Consider the subgroup of the previous group where the product of the nonzero entries in the matrix is 1. This is an index two subgroup of $B_n$, generated by the reflections across $(e_i \pm e_j)^\perp$. It is called $D_n$.

**Collisions of names:** We have $A_1 \cong B_1$, $D_1 \cong \{1\}$, $A_1 \times D_1 \cong I_2(1)$, $A_1 \times A_1 \cong D_2 \cong I_2(2)$, $A_2 \cong I_2(3)$, $B_2 \cong I_2(4)$ and $A_3 \cong D_3$. (The last is not obvious.) Some authors will claim that some of these notations are not defined, but if you define them in the obvious ways, this is what you have. Also, $I_2(6)$ has another name, $G_2$.

We have now listed all but finitely many of the finite orthogonal reflection groups. The remaining cases are probably best understood after we start proving the classification, but we'll try to say something about them. Their names are $E_6$, $E_7$, $E_8$, $F_4$, $H_3$ and $H_4$.

**Sporadic regular solids:** $H_3$ is the symmetry group of the dodecahedron. $H_4$ is the symmetry group of a regular 4-dimensional polytope called the 120-cell, which has 120 dodecahedral faces.

**Symmetries of lattices** $F_4$ is the group of symmetries of the lattice $\mathbb{Z}^4 \cup \left[\mathbb{Z}^4 + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right]$ in $\mathbb{R}^4$. $E_8$ is the group of symmetries of the eight dimensional lattice

$$\left\{(a_1, a_2, \ldots, a_8) \in \mathbb{Z}^8 \cup \left[\mathbb{Z}^8 + \left(\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}\right)\right] : \sum a_i \in 2\mathbb{Z}\right\}.$$

$E_7$ is the subgroup of this stabilizing $(1,1,1,1,1,1,1,1)$; $E_6$ is the subgroup of $E_7$ stabilizing $(0,0,0,0,0,0,1,1)$.

**September 8 – Arrangements of hyperplanes.** Let $V$ be a finite dimensional $\mathbb{R}$-vector space and let $H_1, H_2, \cdots, H_N$ be finitely many hyperplanes in $V$. Let $H_i = \beta_i^\perp$ for some vectors $\beta_i \in V^\vee$. The $H_i$ divide $V$ up into finitely many polyhedral cones.

Choose some $\rho$ not in any $H_i$. Let $D^0$ be the open polyhedral it lies in and $D$ its closure. We'll choose our normal vectors $\beta_i$ such that $\langle \beta_i, \rho \rangle > 0$. Observe: There can be no linear relation between the $\beta$s of the form $\sum c_i \beta_i = 0$ with $c_i \geq 0$ (except $c_1 = c_2 = \cdots = c_N = 0$) since then $\sum c_i \langle \beta_i, \rho \rangle = 0$.

We will call $\beta_i$ **simple** if $\beta_i$ is **not** of the form $\sum_{j \neq i} c_j \beta_j$ with $c_j \geq 0$. So

$$D = \{\sigma : \langle \beta_i, \sigma \rangle \geq 0, 1 \leq i \leq n\} = \{\sigma : \langle \beta_i, \sigma \rangle \geq 0, \beta_i \text{ simple }\}.$$

In Figure 1, $\beta_1$ and $\beta_2$ are simple and $\beta_3$ is not.

Now let $H_i$ be the reflecting hyperplanes of some finite orthogonal reflection group. (Now $V$ has inner product) Let $\rho, D^0, D$ as before, $\beta_i$ as before. Call the $\beta_i$ "positive roots". Let $\alpha_1, \cdots, \alpha_k$ be the simple roots.

**Example.** Consider $A_{n-1}$ (the symmetric group $S_n$) acting on $\mathbb{R}^n$. The reflections are the transpositions $(ij)$; the hyperplanes are $x_i = x_j$ with normal vectors $e_i - e_j$. If we take $\rho = (1, 2, \cdots, n)$, then the positive roots are $e_i - e_j$ for $(i > j)$.

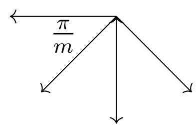
The simple roots are $e_{i+1} - e_i =: \alpha_i$; for anything else, $e_j - e_i = (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \cdots + (e_{i+1} - e_i)$. So $D = \{(x_1, x_2, \cdots, x_n) : x_1 \leq x_2 \leq \cdots \leq x_n\}$.

What's the angle between $\alpha_1$ and $\alpha_3$? Since $\langle \alpha_1, \alpha_3 \rangle = 0$, it is $\frac{1}{2}\pi$.

What's the angle between $\alpha_1$ and $\alpha_2$? We have $\langle \alpha_1, \alpha_1 \rangle = (1)^2 + (-1)^2 = 2$, so $|\alpha_1| = \sqrt{2} = |\alpha_2|$. $\alpha_1 \cdot \alpha_2 = (e_2 - e_1) \cdot (e_3 - e_2) = -1$, so $\cos\theta = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2}$ and $\theta = \frac{2}{3}\pi$.

**Lemma.** In any finite reflection group with $\alpha_1, \cdots, \alpha_k$ as before, the angle between $\alpha_i$ and $\alpha_j$ is of the form $\pi(1 - \frac{1}{m_{ij}})$, $m_{ij} \in \{2, 3, 4, \cdots\}$. Letting $s_i$ be reflection over $\alpha_i^\perp$ the integer $m_{ij}$ is the order of $s_i s_j$.

*Proof.* Let $\theta_{ij}$ be the angle between $\alpha_i$ and $\alpha_j$. Then $s_i s_j$ is rotation by $\theta_{ij}$ around the axis $\alpha_i^\perp \cap \alpha_j^\perp$. Letting $m_{ij}$ be order of $s_i s_j$, we see $\theta_{ij}$ must be of the form $\frac{l_{ij}}{m_{ij}}\pi$, where $\mathrm{GCD}(l_{ij}, m_{ij}) = 1$. So $s_i, s_j$ generate a copy of the dihedral group of order $2m_{ij}$. The $m_{ij}$ reflections in that subgroup are in mirrors $\frac{\pi}{m_{ij}}$ apart. So the corresponding positive roots look like this:




The simple roots $\alpha_i$ and $\alpha_j$ must be the two at the ends, as the other roots aren't simple. □

**Corollary.** For $i \neq j$, $\langle \alpha_i, \alpha_j \rangle \leq 0$.

*Proof.* $\cos \pi(1 - \frac{1}{m}) \leq 0$ for $m \geq 2$. □

**Lemma.** $\alpha_1, \alpha_2, \cdots, \alpha_k$ are linearly independent.

*Proof.* Suppose $\sum c_i \alpha_i = 0$. We already know can't have all $c_i \geq 0$. We also can't have all $c_i \leq 0$.

Rewrite the supposed relation as $\sum_{i \in I} c_i \alpha) i - \sum_{j \in J} d_j \alpha_j = 0$ where $I \cap J = \emptyset$ and $c_i$, $d_j > 0$.

Then $\sum_{i \in I} c_i \alpha_i = \sum_{j \in J} d_j \alpha_j$, we call this sum $\gamma$. But then $\gamma \cdot \gamma = \sum_{i \in I, j \in J} c_i d_j \langle \alpha_i, \alpha_j \rangle \leq 0$. So $\gamma = 0$, and we already ruled that out. □

Next time, we will classify all tables of $m_{ij}$ such that there exist $\alpha_1, \ldots, \alpha_k$ in $\mathbb{R}^k$, a basis, with angle $\pi(1 - m_{ij})$ between $\alpha_i$ and $\alpha_j$.

**September 11 – Classification of positive definite Cartan matrices.** We review the constructions from last time: Let $V$ be a finite dimensional vector space with an inner product. Let $W \subseteq GL(V)$ be a finite orthogonal reflection group and $H_1, \ldots, H_n$ be the hyperplanes corresponding to the reflections in $W$. Choose $\rho \in V - \bigcup_i H_i$. Let $\beta_i$ be normal to $H_i$ such that $\langle \beta_i, \rho \rangle > 0$ and we let $\alpha_1, \ldots, \alpha_k$ be the simple roots of this. Let $s_i$ be the reflection in $\alpha_i^\perp$.

Then, we showed last class that the $\alpha_i$ are linearly independent and the angle between $\alpha_i$ and $\alpha_j$ is $\pi\left(1 - \frac{1}{m_{ij}}\right)$, where $m_{ij}$ is the order of $s_i s_j$.

**<u>Main Idea:</u>** Today, we will forget about the reflection group and look at sets of vectors with these properties. It turns out that this is a very limited set.

We begin by normalizing $|\alpha_i|$ to $\sqrt{2}$ so $\langle \alpha_i, \alpha_j \rangle = 2\cos\left[\pi\left(1 - \frac{1}{m_{ij}}\right)\right] = -2\cos\frac{\pi}{m_{ij}}$. We call this $A_{ij}$. We also have $A_{ii} = \langle \alpha_i, \alpha_i \rangle = 2$. $A$ then forms a symmetric matrix.

**Example.** Let $W = S_n = A_{n-1}$. So $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$. We have

$$A = \begin{bmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{bmatrix}.$$

**Proposition.** $A$ is positive definite.

*Proof.* Let $0 \neq c \in \mathbb{R}^k$. Then, $cAc^T = \sum_{i,j} c_i c_j \langle \alpha_i, \alpha_j \rangle = \left\langle \sum_i c_i \alpha_i, \sum_j c_j \alpha_j \right\rangle \geq 0$. There is equality here iff $\sum c_i \alpha_i = 0$, but as $\alpha_i$ linearly independent and $c \neq 0$, this doesn't happen. Thus, $cAc^T > 0$ . $\square$

We encode $A$ in a graph $\Gamma$: The vertices are $1, \ldots, k$ and we include an edge $(i, j)$ if $m_{i,j} \geq 3$ (so $A_{ij} < 0$) and we label those edges with $m_{ij}$ if $m_{ij} > 3$. These are called ***Coxeter diagrams***. Since positive definite matrices have all submatrices positive definite, $A_G$ cannot contain a subgraph corresponding to a non-positive definite matrix. We will thus begin to list non-positive definite graphs. To do this, we use the following convention: If a matrix $S$ isn't positive-definite, there is a nonzero vector $c$ such that $cSc^T \leq 0$. We will label the vertex $i$ with $c_i$ in the below graphs to illustrate why each graph is not postive definite.

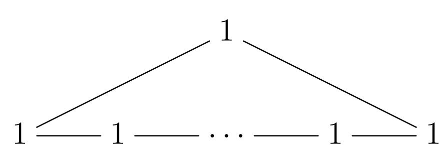
We'll classify the connected Coxeter diagrams. We will show later that, if $W$ gives rise to a Coxeter diagram $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then $W$ breaks up as $W_1 \times W_2$ correspondingly.

Before we begin, we remind the reader of a few key values:

| $m_{ij}$ | $A_{ij}$ |
|---|---|
| 2 | 0 |
| 3 | $-1$ |
| 4 | $-\sqrt{2}$ |
| 5 | $-\frac{1+\sqrt{5}}{2}$ |
| 6 | $-\sqrt{3}$ |

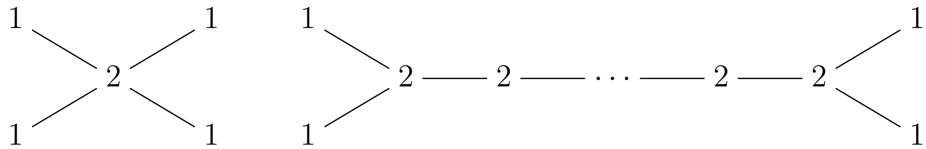
Note that the following subgraphs are excluded:

(1)




This implies $\Gamma$ is a tree.

(2)

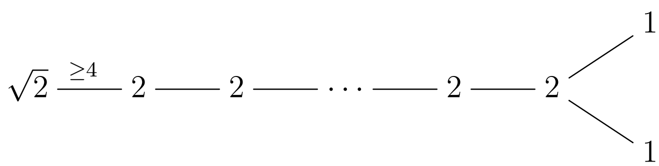



This implies Γ has no vertices of degree ≥ 4, and at most one of degree 3.

(3)


This implies Γ has at most one edge with $m_{ij} \geq 4$.

(4)

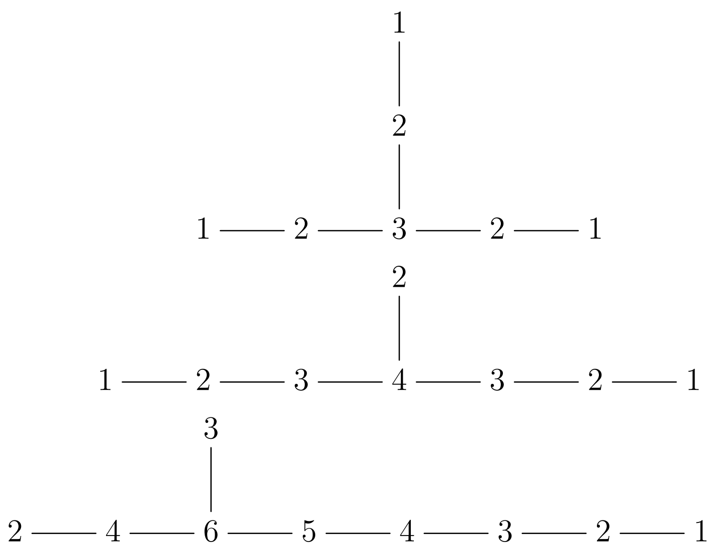



This implies Γ does not have both an edge with $m_{ij} \geq 4$ and a vertex of degree 3.

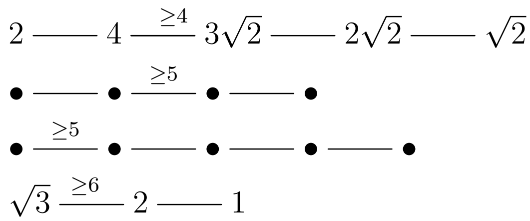
At this point, we know that Γ is either

(1) 3 paths with a common endpoint, and all edges with $m_{ij} = 3$ or
(2) A single path, with at most one edge having $m_{ij} > 3$.

The following excluded graphs rule out all but finitely many three path cases:



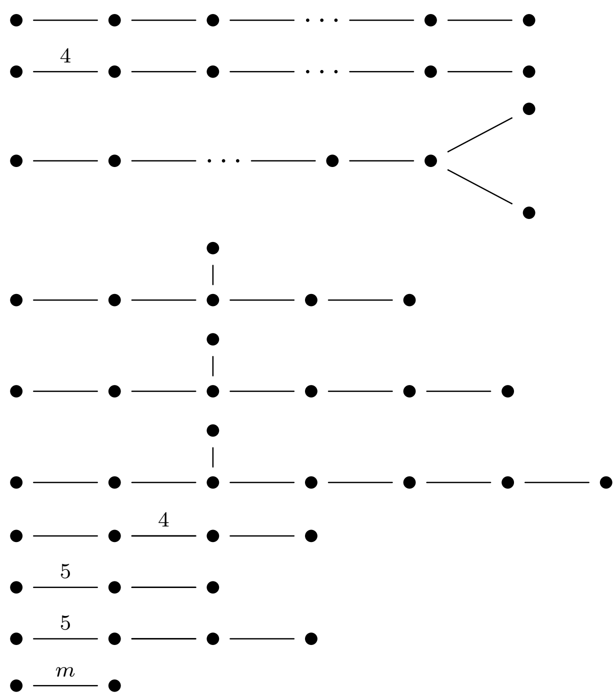

The following excluded graphs rule out all but finitely many single path cases. In the middle two cases, there is no particularly natural choice of vector with negative inner product, so we just compute the determinant.




det $= 3 - 2\sqrt{5} < 0$

det $= 4 - 2\sqrt{5} < 0$

The surviving graphs are the (connected) Coxeter diagrams! Recall that unlabeled edges now mean $m_{ij} = 3$.




$A_n$

$B_n$

$D_n$

$E_6$

$E_7$

$E_8$

$F_4$

$H_3$

$H_4$

$I_2(m)$

There are also some alternate names:

$$B_n = C_n \quad A_2 = I_2(3) \quad B_2 = I_2(4) \quad G_2 = I_2(6) \quad A_3 = D_3.$$

**Reversing the Process:** We have now shown that every reflection group generates one of these Coxeter diagrams, but in order to conclude that this is a classification, we need to show that this process can be reversed and is a bijection. It is clear that we can go from the graph $\Gamma$ to the matrix $A$. Then, we have the following:

**Theorem.** For any positive definite symmetric matrix $A$, there exist vectors $\alpha_1, \alpha_2, ..., \alpha_n$ that are linearly independent such that $\langle \alpha_i, \alpha_j \rangle = A_{ij}$.

*Proof.* Write $A = UDU^T$ with $U$ orthogonal, $D$ diagonal. Then, $D = \operatorname{diag}(\lambda_1, ..., \lambda_n)$. Let $X = U \cdot \operatorname{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$. Then, $A = XX^T$ and the rows of $X$ have the desired property. $\square$

We can thus go from $A$ to some $\alpha_1, \ldots, \alpha_k$, with corresponding reflections $s_i(x) = x - 2\frac{\langle \alpha_i, x\rangle}{\langle \alpha_i, \alpha_j \rangle}\alpha_i$. These then generate some group $W_{\text{output}}$.

**However:** The following issues still remain

(1) Is $W_{\text{output}}$ necessarily finite?

(2) If we start with a group $W_{\text{input}}$, run the first algorithm to get a graph $\Gamma$ and then run it backwards to get $W_{\text{output}}$, does $W_{\text{output}} = W_{\text{input}}$? (Note: If this is yes, it deals with the issue of non-connected graphs from before)
(3) If we change $\rho$, do we always get the same Coxeter diagram $\Gamma$?
(4) If we start with $\alpha_i, \ldots, \alpha_k$, build $W_{\text{output}}$ and take a $\rho$ with $\langle \rho, \alpha_i \rangle > 0$, will $\alpha_i, \ldots, \alpha_k$ be the simples we get?

We will show on October 2 that the answer to all these questions is yes.

**September 13 – Non-orthogonal reflections.** Let $V$ be a finite dimensional $\mathbb{R}$ vector space, $V^\vee$ its dual. We call an element $\sigma \in \mathrm{GL}(V)$ is a reflection if $\sigma^2 = id$ and $V^G$ is of codimension one. Then reflections look like $\sigma(x) = x - \langle \alpha^\vee, x \rangle \alpha$ for some $\alpha \in V$, $\alpha^\vee \in V^\vee$ with $\langle \alpha^\vee, \alpha \rangle = 2$. Here $\alpha$ and $\alpha^\vee$ are only determined up to rescaling of the form

$$\alpha \mapsto c\alpha, \alpha^\vee \mapsto c^{-1}\alpha^\vee.$$

Up until now, we had a dot product (a symmetric bilinear form) on $V$, giving $V \cong V^\vee$. In this setting, $\sigma$ is orthogonal if $\alpha$ and $\alpha^\vee$ are proportional. Then we will have $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

A **reflection group** is defined as a subgroup of $\mathrm{GL}(V)$ generated by reflections.

**Lemma.** If $G$ is a finite subgroup of $\mathrm{GL}(V)$, then $G$ preserves a positive definite symmetric bilinear form.

*Proof.* Take any positive-definite bilinear form ( , ). Set

$$\langle \vec{v}, \vec{w} \rangle := \frac{1}{G} \sum_{g \in G} (g\vec{v}, g\vec{w}).$$

Then $\langle\ ,\ \rangle$ is positive definite and $G$-invariant. □

So all finite reflection groups are orthogonal.

**Two reflections** We now look at the case of two reflections. Let $\sigma$ and $\tau$ be two reflections such that

$$\sigma(x) = x - \langle \alpha^\vee, x \rangle \alpha, \tau(x) = x - \langle \beta^\vee, x \rangle \beta,$$

with $\langle \alpha^\vee, \alpha \rangle = \langle \beta^\vee, \beta \rangle = 2$. We will care about the case when $\langle \beta^\vee, \alpha \rangle$, $\langle \alpha^\vee, \beta \rangle \le 0$, and $\alpha, \beta$ are linearly independent. Then the action on $Span(\alpha^\vee, \beta^\vee)^\perp$ is trivial.

Now we consider the action on $Span(\alpha, \beta)$. Since $\sigma(\alpha) = -\alpha$, and $\sigma(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$, under the $(\alpha, \beta)$-basis, we have

$$\sigma = \begin{bmatrix} -1 & -\langle \alpha^\vee, \beta \rangle \\ 0 & 1 \end{bmatrix} \ \tau = \begin{bmatrix} 1 & 0 \\ -\langle \beta^\vee, \alpha \rangle & -1 \end{bmatrix}$$

and

$$\sigma\tau = \begin{bmatrix} -1 + \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle & \langle \alpha^\vee, \beta \rangle \\ -\langle \beta^\vee, \alpha \rangle & -1 \end{bmatrix}$$

$$det(\sigma\tau) = 1, Tr(\sigma\tau) = \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle - 2.$$

Since we have $\langle \beta^\vee, \alpha \rangle$, $\langle \alpha^\vee, \beta \rangle \le 0$, we know that $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \ge 0$.

Case I Assume $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle \in [0, 4)$. We set $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle = 4\cos^2\theta$. Then we have $Tr(\sigma\tau) = 4\cos^2\theta - 2 = 2\cos 2\theta$. So eigenvalues of $\sigma\tau$ are $e^{\pm i 2\theta}$. This is the rotation by $2\theta$. So the group $< \sigma, \tau >$ is dihedral of order $2m$ if $\theta = \frac{\pi l}{m}$, with $\gcd(l, m) = 1$. Otherwise, it will become infinite.

Case II When $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \ge 4$, then the group $< \sigma, \tau >$ is infinite dihedral.

**Coxeter group** Now let $m_{ij}$ be in $\{1,\ldots,\infty\}$, for $1 \le i,j \le k$. We set $m_{ii}=1$, and $m_{ij}=m_{ji}\ge 2$. The Coxeter group $W$ defined by those $m's$ is

$$< s_i|(s_is_j)^{m_{ij}} = 1 > .$$

We call a group which can be presented by this way a Coxeter group. Those $s_i$ are called the simple generators.

**Cartan matrix** A matrix $A$ is called a Cartan matrix for $m_{ij}$ if we have

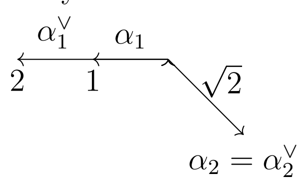
$$\begin{array}{rcllr} A_{ii} & = & 2 & & \\ A_{ij} & = & 0 & & m_{ij}=2 \\ A_{ij}A_{ji} & = & 4\cos^2\frac{\pi}{m_{ij}}, & A_{ij},A_{ji}<0 & 3\le m_{ij}<\infty \\ A_{ij}A_{ji} & \ge & 4, & A_{ij},A_{ji}<0 & m_{ij}=\infty \end{array}$$

Note that $A_{ij}$ and $A_{ji}$ do not need to be equal. We then say two sets $\alpha_1,\ldots,\alpha_k\in V$ and $\alpha_1^\vee,\ldots,\alpha_k^\vee\in V^\vee$**pair by** $A$ if

$$\langle\alpha_i^\vee,\alpha_j\rangle = A_{ij}.$$

Then we define $s_i : V\to V$ as $s_i(x)=x-\langle\alpha_i^\vee,x\rangle\alpha$, which gives an action of $W$ on $V$.

**Example.** Consider the dihedral order $B_2=I_2(4)$. Then its hyperplanes and decomposition of complements are given by a two dimensional plane cut out by $y\pm x=0$. The associated simple roots and their dual are given by




From the diagram, we see the Cartan matrix is

$$A=\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix},$$

the reflection with respect to hyperplanes are

$s_1=\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, s_2=\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. Warning: These formulas were given wrongly in class!

From this, we get an example of a Cartan matrix coming from a Coxeter group which is not symmetric. In this case, we could instead choose to take $\alpha_1=\alpha_1^\vee$, both of length $\sqrt{2}$. In infinite groups, there are examples which cannot be made symmmetric.

**September 15 – Roots, hyperplanes, length, reduced words.** Last time, we discussed the data needed for a Coxeter group; namely, a collection of $m_{ij}$'s, with $m_{ij}\in\{1,2,3,\ldots,\infty\}$ so that $m_{ii}=1$, and $m_{ij}=m_{ji}\ge 2$. This will define the Coxeter group $W=\langle s_i|(s_i,s_j)^{m_{ij}}=1\rangle$.

A Cartan matrix, $A$, for our Coxeter group $W$ was then defined by $A_{ij}$ satisfying:

$$\begin{aligned} A_{ii} &= 2, & \\ A_{ij} &= 0 & \text{if } m_{ij} = 2, \\ A_{ij}A_{ji} &= 4\cos^2 \tfrac{\pi}{m_{ij}},\ A_{ij}, A_{ji} < 0 & \text{if } 3 \leq m_{ij} < \infty, \text{ and} \\ A_{ij}A_{ji} &\geq 4,\ A_{ij}, A_{ji} < 0 & \text{if } m_{ij} = \infty. \end{aligned}$$

If $V$ and $V^\vee$ are dual vector spaces with $\alpha_1, \alpha_2, \ldots, \alpha_k \in V$ and $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_k^\vee \in V^\vee$, we say that the pair by $A$ if $\langle \alpha_i^\vee, \alpha_j \rangle = A$. In this case, we get actions on both $V$ and $V^\vee$ with $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$, and $s_i(x^\vee) = x^\vee - \langle x^\vee, \alpha_i \rangle \alpha_i^\vee$.

**Example.** In question 3 on homework 1, we played with: $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. If $\alpha_1$, $\alpha_2$, are a basis for $V$, and $w_1$, $w_2$ are a dual basis for $V^\vee$:

$$\alpha_1^\vee = 2w_1 - 2w_2 \text{ and } \alpha_2^\vee = -2w_1 + 2w_2.$$

In particular, the $\alpha_i^\vee$ do not necessarily form a basis for $V^\vee$.

Let $D = \{x^\vee \in V^\vee | \langle x^\vee, \alpha_i \rangle \geq 0\}$, and $D^\circ = \{x^\vee \in V^\vee | \langle x^\vee, \alpha_i \rangle > 0\}$. Generally, we want $D^\circ$ non-empty. (As an example of what this forbids, when $A = \left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$, we don't want $V \cong V^\vee \cong \mathbb{R}$ with $\alpha_1 = \alpha_1^\vee = \sqrt{2}$ and $\alpha_2 = \alpha_2^\vee = -\sqrt{2}$.) We already have our simple roots - let's get the rest. Let $\Phi = W \cdot \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset V$. In this much generality, we could have $\beta$ and $c\beta$ for some $c \in \mathbb{R} \setminus \{\pm 1\}$ both in $\Phi$.

Each $\beta \in \Phi$ is of the form $w\alpha_i$ for some $w \in W$, and some $i$. We have $ws_iw^{-1}$, which is a reflection acting by $x \mapsto x - \langle \beta^\vee, x \rangle \beta$, with $\beta^\vee = w\alpha_i^\vee$. This will turn out to be all of the reflections!

**Lemma.** There is a homomorphism sgn : $W \longrightarrow \pm 1$ given by:

$$s_i \longmapsto -1.$$

*Proof.* Recall $W = \langle s_i | (s_i, s_j)^{m_{ij}} = 1 \rangle$. All relations are sent to 1 if we send $s_i$ to $-1$. □

*Proof (alternate).* Choose a cartan matrix, roots, and coroots. This gives us:

$$\begin{array}{ccc} W & \longrightarrow & GL(V) \\ \downarrow{\scriptstyle \text{sgn}} & & \downarrow{\scriptstyle \det} \\ \{\pm 1\} & \hookrightarrow & \mathbb{R}^* \end{array}$$

□

**The Length Function:** For $w \in W$, define $\ell(w)$ to be the minimal $\ell \in \mathbb{Z}_{\geq 0}$ such that we can write $w = s_{i_1} s_{i_2} \ldots s_{i_\ell}$ for some sequence of simple reflections. Notice $\ell(w) = 0$ if and only if $w = \text{Id}$. We also have $\text{sgn}(w) = (-1)^{\ell(w)}$, so $\ell(w)$ isn't a homomorphism, but it lifts sgn to non-negative integers.

We define a word $(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell}$ to be **reduced** if $\ell = \ell(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})$. A (consecutive) subword of a reduced word is reduced, i.e. if $s_1 s_2 \ldots s_\ell$ is reduced, $s_i s_{i+1} \ldots s_{i+j}$ is reduced.

**Lemma.** For $w \in W$ and any simple generator $s_i$, $\ell(s_i w) = \ell(w) \pm 1$ and $\ell(w s_i) = \ell(w) \pm 1$.

*Proof.* We have $\ell(s_i w) \leq \ell(w)+1$ and $\ell(w) \leq \ell(s_i w)+1$. So, $\ell(s_i w) - \ell(w) \in \{-1,0,1\}$. We also have

$$(-1)^{\ell(w)} = \operatorname{sgn}(w) = -\operatorname{sgn}(s_i w) = -(-1)^{\ell(s_i w)},$$

so $\ell(w) \equiv \ell(s_i w) + 1 \mod 2$. □

We will say $s_i$ is a **left ascent/descent of** $w$ according to whether:

$$\ell(s_i w) = \ell(w) + 1 \text{ (left ascent), or}$$
$$\ell(s_i w) = \ell(w) - 1 \text{ (left descent).}$$

We also characterize right ascent/descent, with

$$\ell(w s_i) = \ell(w) + 1 \text{ (right ascent), and}$$
$$\ell(w s_i) = \ell(w) - 1 \text{ (right descent).}$$

Any $w$ other than the identity has some left descent and some right descent.

Note: This terminology makes sense in $S_n$. In $S_n$,

$$\ell(w) = \#\{(i,j) | 1 \leq i < j \leq n, w(i) > w(j)\}.$$

These are inversions. We get $s_i$ is a left ascent if $w^{-1}(i) < w^{-1}(i+1)$, and is a right ascent if $w(i) < w(i+1)$.

**Example.** Let 231 be $2 \mapsto 1$, $3 \mapsto 2$, $1 \mapsto 3$. Then,

- $s_2$ is a right descent of $2\underline{31}$, because *positions* 2 and 3 are out of order, and
- $s_2$ is a left ascent of $\underline{23}1$, because *numbers* 2 and 3 are in order.

**September 18 – Rank two computations.** On the homework, you were asked to describe the rank two reflection groups coming from Cartan matrices $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right]$. These will be key computations, so we repeat them here. The notes contain some material not cover in class.

Let $V$ have basis $\alpha_1$, $\alpha_2$ and let $\omega_1$, $\omega_2$ be the dual basis of $V^\vee$. If we want $\alpha_i^\vee$ to pair with $\alpha_j$ by $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$, we should take

$$\alpha_1^\vee = 2\omega_1 - 2\omega_2 \qquad \alpha_2^\vee = -2\omega_1 + 2\omega_2.$$

Note that $s_1^2 = s_2^2 = 1$ so every element of $W$ can be expressed by a word which alternates $s_1$ and $s_2$. We compute

$$\begin{array}{rcrcr} \alpha_1 & = & \alpha_1 & & \\ s_2\alpha_1 & = & \alpha_1 & + & 2\alpha_2 \\ s_1 s_2 \alpha_1 & = & 3\alpha_1 & + & 2\alpha_2 \\ s_2 s_1 s_2 \alpha_1 & = & 3\alpha_1 & + & 4\alpha_2 \end{array}.$$

In general $(s_1 s_2)^k \alpha_i = (2k+1)\alpha_1 + (2k)\alpha_2$. Similar computations show that

$$W\{\alpha_1, \alpha_2\} = \{n\alpha_1 + (n \pm 1)\alpha_2\}.$$

So the roots $\Phi$ lie on two parallel lines. The reflections $s_1$ and $s_2$ both fix the line $\mathbb{R}(\alpha_1 + \alpha_2)$ but they move elements parallel to $\alpha_1$ and $\alpha_2$ respectively.

Let $D = \{x^\vee \in V^\vee : \langle x^\vee, \alpha_i \rangle \geq 0\}$ for $i = 1$, 2. So, in $\omega$ coordinates, this is the first quadrant. Then

$$wD = \{x^\vee \in V^\vee : \langle w^{-1} x^\vee, \alpha_i \rangle \geq 0\} = \{x^\vee \in V^\vee : \langle x^\vee, w\alpha_i \rangle \geq 0\}.$$

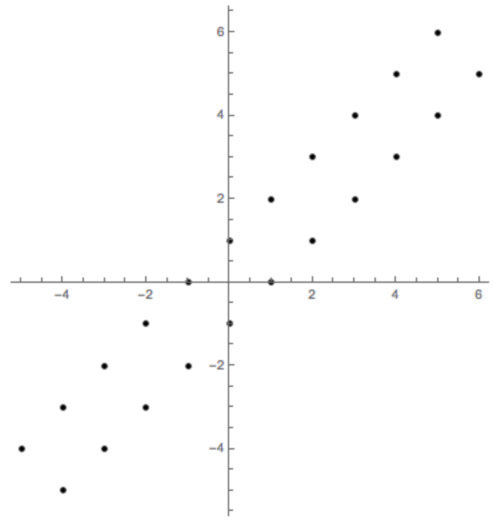



FIGURE 2. The roots for Cartan matrix $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$

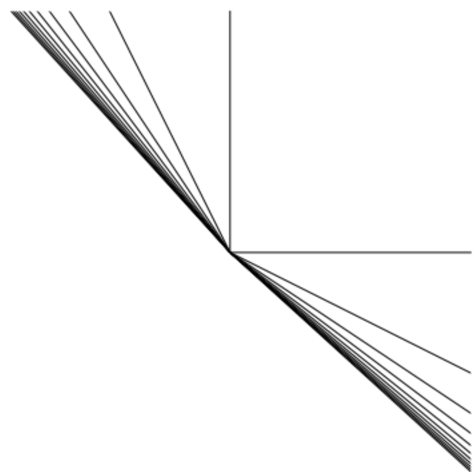


FIGURE 3. The regions $wD$ for Cartan matrix $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$

For example,

$$\begin{aligned} s_1 D &= \{x^\vee : \langle x^\vee, s_1\alpha_1\rangle,\ \langle x^\vee, s_1\alpha_2\rangle \geq 0\} \\ &= \{x^\vee : \langle x^\vee, -\alpha_1\rangle,\ \langle x^\vee, 2\alpha_1+\alpha_2\rangle \geq 0\} \\ &= \{x_1\omega_1 + x_2\omega_2 : x_1 \leq 0,\ 2x_1 + x_2 \geq 0\} \end{aligned}$$

In general, the $wD$ form a collection of wedges filling up the half plane $x_1 + x_2 = 0$:

The $wD$ and $-wD$ together fill up $V^\vee$. The same qualitative behavior (filling a half plane) occurs for $\left[\begin{smallmatrix} 2 & -b \\ -c & 2 \end{smallmatrix}\right]$ whenever $bc = 4$.

**Remark.** We will prove later that $-D \in \bigcup_{w\in W} wD$ if and only if $W$ is finite.

**Remark.** This example is sometimes called "the infinite dihedral group". Indeed, if we write the $I_2(m)$ root system in the basis $\alpha_1$, $\alpha_2$, then the roots are

$$\left(\sin\tfrac{\pi k}{m}\alpha_1 + \sin\tfrac{\pi(k-1)}{m}\alpha_2\right) / \sin\tfrac{\pi}{m} \quad k \in \mathbb{Z}.$$

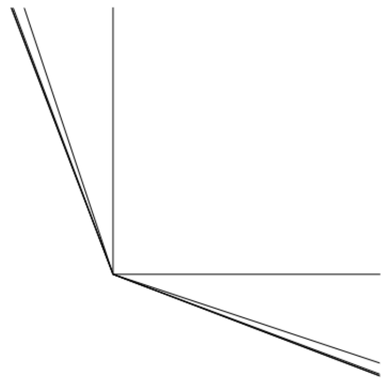


FIGURE 4. The regions $wD$ for Cartan matrix $\left[\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right]$

These lie on an ellipse and, as $m \to \infty$, the ellipse approaches two parallel lines with points at $(k, k \pm 1)$.

If we instead work with the Cartan matrix $\left[\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right]$, we now compute that

$$W\{\alpha_1, \alpha_2\} = \{F_{2k}\alpha_1 + F_{2k\pm 2}\alpha_2\}$$

where $F_i$ are the Fibonacci numbers (indexed as $(F_0, F_1, F_2, F_3, F_4, \ldots) = (0, 1, 1, 2, 3, \ldots)$. So the regions $wD$ now have rays with slope $-F_{2k}/F_{2k\pm 2}$. In the limit, this slope approaches $-\tau^2 \approx -2.618\cdots$ where $\tau$ is the Golden Ratio. So the $wD$ only fill up a convex wedge, and must of $V^\vee$ is in neither $\bigcup wD$ nor $-\bigcup wD$.

The same qualitative behavior occurs for $\left[\begin{smallmatrix} 2 & -b \\ -c & 2 \end{smallmatrix}\right]$ whenever $bc > 4$. Indeed, the eigenvalues of $s_1 s_2$ are $e^{\pm 2\sigma}$ where $bc = 4\cosh^2\sigma$ and the eigenlines always have slope $-e^{\pm 2\sigma}$ in the $\omega_1$, $\omega_2$ coordinates. The region $\bigcup_{w\in W} wD$ fills in a convex wedge bounded by the eigenlines.

**September 20 – A key lemma.** Today's goal is the following key lemma:

**Lemma.** For $w \in W$ and $s_i$ a simple reflection:

- If $s_i$ is a left ascent of $w$, then $\langle \quad , \alpha_i\rangle \geq 0$ on $wD$ and
- If $s_i$ is a left descent of $w$, then $\langle \quad , \alpha_i\rangle \leq 0$ on $wD$ .

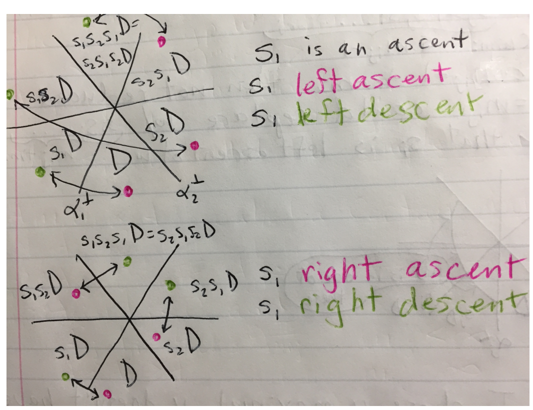



FIGURE 5. Illustration of the lemma for $A_2$

**Remark.** We note that $\alpha_i, \alpha_j$ are not proportional: We can't have $\alpha_i \in \mathbb{R}_{>0}\alpha_j$ as $\langle \alpha_i^\vee, \alpha_i \rangle = 2$ and $\langle \alpha_i^\vee, \alpha_j \rangle \leq 0$, and we can't have $\alpha_i \in \mathbb{R}_{<0}\alpha_j$ as that would make $D^0 = \emptyset$.

*Proof.* We proceed by induction on $\ell(w)$

Base case: $\ell(w) = 0$ so $w = 1$. We want to show $\langle \quad, \alpha_i \rangle \geq 0$ on $D$. This is true by the definition of $D$.

Case I: $s_i$ is a left descent of $w$ so $\ell(s_i w) = \ell(w) - 1$. Inductively, $\langle \alpha_i, x \rangle \geq 0$ for $x \in s_i wD$, so $\langle \alpha_i, s_i y \rangle \geq 0$ for $y \in wD$. But $\langle \alpha_i, s_i y \rangle = \langle s_i \alpha_i, y \rangle = -\langle \alpha_i, y \rangle$ so $\langle \alpha_i, y \rangle \leq 0$ for $y \in wD$.

Case II: $s_i$ is a left ascent of $w$, but $\ell(w) > 0$. Since $w \neq 1$, there is some left descent $s_j$ of $w$. Choose a reduced word for $w$ of the form $uv$ where $u \in \langle s_i, s_j \rangle$ and is as long as possible. Then $u$ and $v$ are reduced and $\ell(u) + \ell(v) = \ell(w)$. We know $u$ doesn't start with $s_i$ since $s_i$ is a left ascent for $w$, so $u = s_j s_i s_j s_i ... s_{i \text{ or } j}$ We claim that $1 \leq \ell(u) \leq m_{ij} - 1$. If $\ell(u) = 0$ then $s_j$ is not a descent of $w$, contradiction. No element of $\langle s_i, s_j \rangle$ has length greater than $m_{ij}$. Finally, is if $\ell(u) = m_{ij}$ then $u$ could be rewritten as $s_i s_j ...$ with $m_{ij}$ terms, which contradicts $s_i$ being a left ascent of $w$. So the claim is proven.

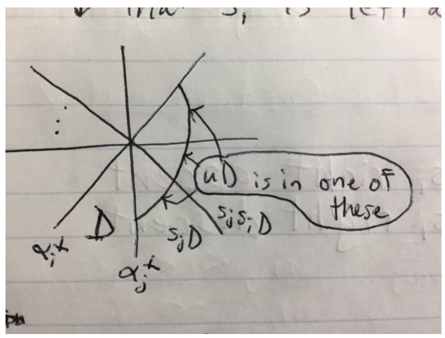



FIGURE 6. Our restrictions on the position of $u$

Since we chose $u$ maximal, $v$ doesn't have reduced words with first letter $s_i$ or $s_j$. So $s_i, s_j$ are both left ascents of $v$. And $\ell(v) = \ell(w) - \ell(u) < \ell(w)$.
So, inductively, $\langle \quad, \alpha_i \rangle \geq 0$ and $\langle \quad, \alpha_j \rangle \geq 0$ on $vD$.
By our results on $u$, we have: $\langle \quad, \alpha_j \rangle \geq 0$ on the set $u\{x | \langle x, \alpha_i \rangle \geq 0, \langle x, \alpha_j \rangle \geq 0\}$ and $vD \subseteq \{x | \langle x, \alpha_i \rangle \geq 0, \langle x, \alpha_j \rangle \geq 0\}$. Therefore, $\langle \quad, \alpha_i \rangle \geq 0$ on $uvD = wD$ ∎

**Corollary.** If $w \in W$ is not 1 then $wD^0 \cap D^0 = \emptyset$

*Proof.* The element $w$ has some left descent $s_i$. We have $\langle \alpha_i, \quad \rangle > 0$ on $wD^0$ and $\langle \alpha_i, \quad \rangle < 0$ on $D^0$ ∎

**Corollary.** $W \to GL(V^\vee)$ is injective. $W \to GL(V)$ is also injective.

**Example.** Let's see what this means for the symmetric group $S_n$. Recall that $(\sigma x)_i = x_{\sigma^{-1}(i)}$ where $\sigma \in S_n$ and $x \in \mathbb{R}^n$. So

$$x \in \sigma D \iff \sigma^{-1} x \in D \iff (\sigma^{-1} x)_i \text{ is increasing} \iff x_{\sigma(1)} \leq x_{\sigma(2)} \leq ... \leq x_{\sigma(n)}.$$

The lemma says that $(i \quad i+1)$ is a left ascent of $\sigma \iff \langle e_{i+1} - e_i, x \rangle \geq 0$ for $x \in \sigma D$. By the above equivalences, this will happen if and only if $\sigma^{-1}(i+1) \geq \sigma^{-1}(i)$.

**September 25 – Consequences of the Key Lemma.** Last time, we proved the Key Lemma:

**Lemma.** We have $\langle \quad, \alpha_i\rangle \leqslant 0$ on $wD$ if $s_i$ is a left ascent of $w$ and we have $\langle \quad, \alpha_i\rangle \geqslant 0$ on $wD$ if $s_i$ is a left descent of $w$.

**Corollary.** For any $w \in W$ other than the identity, we have $wD^\circ \cap D^\circ = \emptyset$.

*Proof.* Let $s_i$ be a descent of $w$. Then $\alpha_i^\perp$ separates $D^\circ$ and $wD^\circ$. □

**Corollary.** For $u, v \in W$, with $u \neq v$, we have $uD^\circ \cap vD^\circ = \emptyset$.

*Proof.* We have $uD^\circ \cap vD^\circ = u(D^\circ \cap u^{-1}vD^\circ)$. □

So all $wD^o$ are disjoint.

**Corollary.** The maps $W \to GL(V)$ and $W \to GL(V^\vee)$ are injective.

We define $\Phi$ to be $\Phi = W \cdot \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset V$.

**Corollary.** For any $\beta \in \Phi$ and $w \in W$, the cone $wD$ lies entirely to one side of $\beta^\perp$.

*Proof.* Let $\beta = u\alpha_i$. We can reduce to $\alpha_i$ and $u^{-1}wD$ and use the Key Lemma. □

**Corollary.** Each $wD^\circ$ is a connected component of $V \setminus \cup_{\beta\in\Phi}\beta^\perp$.

*Proof.* Its enough to show the claim for $D^\circ$. We know $D^\circ$ lies entirely to one side of each $\beta^\perp$, and it is connected, so $D^\circ$ lies in some connected component $E$ of $V \setminus \cup_{\beta\in\Phi}\beta^\perp$. Suppose for contradiction, there is some $x \in E \setminus D^\circ$. Since $x \in D^\circ$, so for each $\alpha_i$, we have $\langle x, \alpha_i\rangle \neq 0$, and for some say $\alpha_j$, have $\langle x, \alpha_i\rangle < 0$. Then $\alpha_j^\perp$ separates $x$ from $D^\circ$. □

We now know that, for any $\beta \in \Phi$ and $w \in W$, the cone $wD$ is entirely to one side of $\beta^\perp$. In particular, for any $\beta \in \Phi$, we have either $\langle \quad, \beta\rangle \geq 0$ or $\langle \quad, \beta\rangle \leq 0$ on $D$. So we can write $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+ = \{x \in V^\vee | \langle x, \beta\rangle > 0\}$ are the positive roots, and $\Phi^- = \{x \in V^\vee | \langle x, \beta\rangle < 0\}$ are the negative roots. We earlier described positive and negative roots using a particular $\rho \in D^\circ$, we now see that the choice of $\rho$ is irrelevant.

The next lemma essentially says that the simple roots are a subset of the $\alpha_i$, but is phrased differently to avoid the question of whether to call a positive scalar multiple of $\alpha_i$ "simple". The graphical intuition is shown in Figure 7.

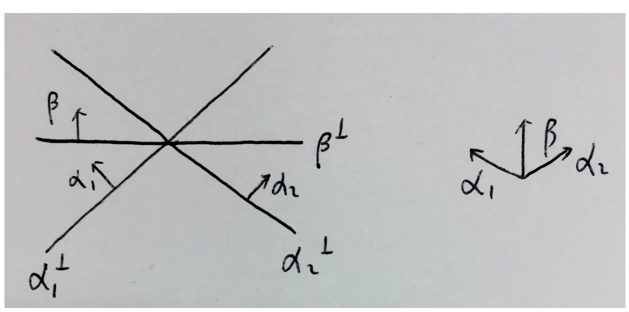



FIGURE 7. $D$ lying on the positive side of $\beta^\perp$ is dual to $\beta$ being in the positive span of the $\alpha_i$

**Lemma.** If $\beta \in \Phi^+$, then $\beta$ can be written as $\sum c_i\alpha_i$ with $c_i \leqslant 0$.

*Proof.* Suppose not, let $R = \{\sum c_i\alpha_i, c_i \geq 0\}$, then $\beta \notin R$, and $R$ is a closed convex set in $V$, so there is some $\gamma \in V^\vee$, with $\langle \gamma, \quad \rangle \geq 0$ on $R$, and $\langle \gamma, \quad \rangle < 0$ on $\beta$. But then $\gamma \in D$, and $\langle \gamma, \beta \rangle < 0$ , its a contradiction. □

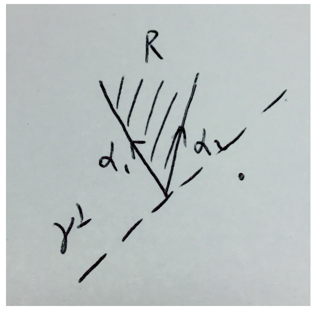



FIGURE 8. Region $R$ and the hyperplane $\gamma^\perp$

**Lemma.** Choose an index $i$. There exists an $\theta \in D$, such that $\langle \theta, \alpha_i \rangle = 0$ and $\langle \theta, \alpha_j \rangle > 0$ for $i \neq j$.

*Proof.* Consider the line segment joining $\rho \in D^\circ$ and $-\alpha_i^\vee$. At $\rho$, all $\langle \quad, \alpha_j \rangle > 0$, and at $-\alpha_i^\vee$, we have $\langle \quad, \alpha_i \rangle = -2$ and $\langle \quad, \alpha_j \rangle \geq 0$ for $i \neq j$. At the point of this segment where $\langle \quad, \alpha_i \rangle = 0$ we must have $\langle \quad, \alpha_j \rangle > 0$ for all $i \neq j$. □

**Corollary.** $\alpha_i \notin \mathrm{Span}_{\mathbb{R}+}\{\alpha_j : \ j \neq i\}$

*Proof.* Let $\alpha_i = \sum_{j \neq i} c_j \alpha_j$, pair with $\theta$ in the previous lemma. □

**Corollary.** There is a nonempty open set of $\alpha_i^\perp$ contained in $D \cap s_i D$

*Proof.* Let U be a small enough neighborhood of $\theta$ in $\alpha_i^\perp$. Then all other $\langle \quad, \alpha_j \rangle > 0$ on $U$ and $\langle \quad, \alpha_i \rangle = 0$ on $U$, so $U \subset D \cap \alpha_i^\perp$. But $s_i D \cap \alpha_i^\perp = s_i(D \cap \alpha_i^\perp) = D \cap \alpha_i^\perp$ since $s_i$ fixes $\alpha_i^\perp$. □

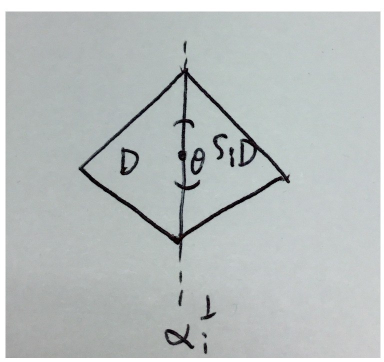



FIGURE 9. The proof of the final corollary

This shows that $D$ and $s_i D$ border along a codimension 1 wall. Thus, for any $v$, the cones $vD$ and $vs_iD$ border along a codimension 1 wall. This raises the following application, which will return next class. Consider a word $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with product $w$. Set $v_k = s_{i_1} s_{i_2} \cdots s_{i_k}$. Then $v_{k-1}D$ and $v_k D = v_{k-1} s_{i_k} D$ border along a codimension one wall. So the sequence of

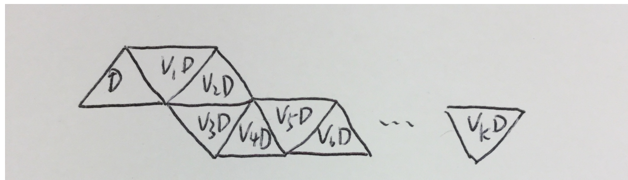



FIGURE 10. A word in $W$ gives a walk through the chambers $wD$

cones $D = v_0 D$, $v_1 D$, $v_2 D$, $\ldots$, $v_\ell D = wD$ each border along codimension one walls, as in Figure 10.

The region $\bigcup_{w \in W} wD$ is called the ***Tits cone*** and denoted $\mathrm{Tits}(W)$. The above argument shows that it is connected in codimension one.

**Correction to class**: Prof. Speyer stated in class that the Tits cone need not be convex. He was wrong, it is always convex. We'll have enough tools to prove this next time.

**September 27 – Adjacency of chambers, the set $T$, assorted homework topics.** Last time:

- For every $w \in W$ and $\beta \in \Phi$, the cone $wD$ lies to one side of the hyperplane $\beta^\perp$
- $\Phi = \Phi^+ \cup \Phi^-$ where $\beta \in \Phi^+$ if $\langle \beta, \ \rangle \geq 0$ on $D$
- $D \cap s_i D$ is an $n-1$-dimensional cone

**Proposition.** For any $\beta \in \Phi$, $\beta$ not a multiple of $\alpha_i$, the cones $D$ and $s_i D$ are on the same side of $\beta^\perp$.

*Proof.* If $D$ and $s_i D$ are on opposite sides of $\beta^\perp$, then $\langle \beta, \rangle = 0$ on $D \cap s_i D$. Then $\mathrm{Span}(D \cap s_i D) = \alpha_i^\perp$ so $\langle \beta, \rangle$ can be 0 on it only if $\beta \in \mathbb{R}\alpha_i$. $\square$

So the chambers bordering $D$ are precisely the chambers $s_i D$. So the chambers bordering $uD$ are $us_i D$. Since $D$ and $s_i D$ are separated by $\alpha_i^\perp$, so their images $uD$ and $us_i D$ are separated by $u\alpha_i^\perp$. So when $uD$ and $vD$ are bordering chambers, they border on $\beta^\perp$ for some $\beta \in \Phi$. In that case $v = us_i$ for some $s_i$ so that $vu^{-1} = us_i u^{-1}$ reflects over that wall.

Let $T \subset W$ be defined as $T = \{us_i u^{-1} : u \in W, \ i = 1, \cdots, n\}$. In $S_n$, these are transpositions.

**Corollary.** If $t_1$ and $t_2 \in T$ reflect over the same hyperplanes, then $t_1 = t_2$.

*Proof.* Let $t_1 = u_1 s_1 u_1^{-1}$ and $t_2 = u_2 s_2 u_2^{-1}$. Consider $t_2 \cdot u_1 D$. Points of $u_1 D^0$ very near the $(u_1 \alpha_1)^\perp$ wall would be mapped by $t_2$ into $t_1 U_1 D^0$. Since the interiors of the chambers are disjoint, $t_2 u_1 = t_1 u_1$ and hence $t_2 = t_1$. $\square$

**From homework:** Problem 3: Given a Cartan matrix $A_{ij}$ and $\alpha_i$ linearly independent, is there some symmetric bilinear form preserved by $W$ with $(\alpha_i, \alpha_i) > 0$? Need there to exist numbers $d_i$ so that $d_i A_{ij} = d_j A_{ji}$. Then the matrix of the inner product is $[d_i A_{ij}]$. If we have such $d_i$, then $A$ is called symmetrizable. In that case, $W$ preserves the inner product $(,)$ so that $|w\alpha_i| = |\alpha_i| = d_i$.

Earlier difficulty: Could we have $c\alpha_i \in \Phi$ for some $c \neq \pm 1$? That is, could we have $w\alpha_j = c\alpha_i$? Then $d_j = \pm c d_i$. Certainly this can't happen if all $d_i$ are equal. If $A$ is symmetrizable, we need to make sure that if $w\alpha_j = \alpha_i$ then $d_j = d_i$. The hyperplane $(w\alpha_j)^\perp$ is fixed by $ws_j w^{-1}$ and $\alpha_i^\perp$ is fixed by $s_i$. So if $w\alpha_j = \alpha_i$ then $ws_j w^{-1} = s_i$. We don't have to worry; assuming $d_i = d_j$ when $s_i$ and $s_j$ are conjugate in $W$. Problem 2 classifies when $s_i$

and $s_j$ are conjugate to each other. Namely, erase all edges of $\Gamma$ with even labels (where $\infty$ is even). Then $s_i$ and $s_j$ are conjugate if and only they are in the same component.

**An action of the affine symmetric group (Problem 4):** We defined $V^\vee = \{(a_i)_{i\in\mathbb{Z}} : a_{i+n} = pa_i\}$ where $p > 1$. Take $n = 3$ for example. Then

$$s_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ s_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ s_3 = \begin{bmatrix} 0 & 0 & p \\ 0 & 1 & 0 \\ p^{-1} & 0 & 0 \end{bmatrix}.$$

We compute

$$\mathrm{Id} - s_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathrm{Id} - s_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix},\ \mathrm{Id} - s_3 = \begin{bmatrix} 1 & 0 & -p \\ 0 & 0 & 0 \\ -p^{-1} & 0 & 1 \end{bmatrix}$$

so we can take

$$\alpha_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\ \alpha_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\ \alpha_3 = \begin{bmatrix} 1 & 0 & -p \end{bmatrix}$$
$$\alpha_1^\vee = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\ \alpha_2^\vee = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\ \alpha_3^\vee = \begin{bmatrix} 1 \\ 0 \\ -p^{-1} \end{bmatrix}.$$

The Cartan matrix is thus

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -p \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -p^{-1} & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

which is clearly not diagonalizable. Indeed, the action preserves no norm and you can check that $(s_1s_2s_3)^2\alpha_2 = p\alpha_2$. The Tits cone in this case is an orthant, as shown in Figure 11.

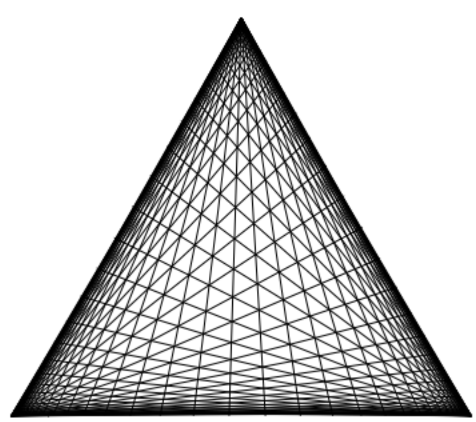


FIGURE 11. The representation of $\tilde{A}_2$ discussed above

.

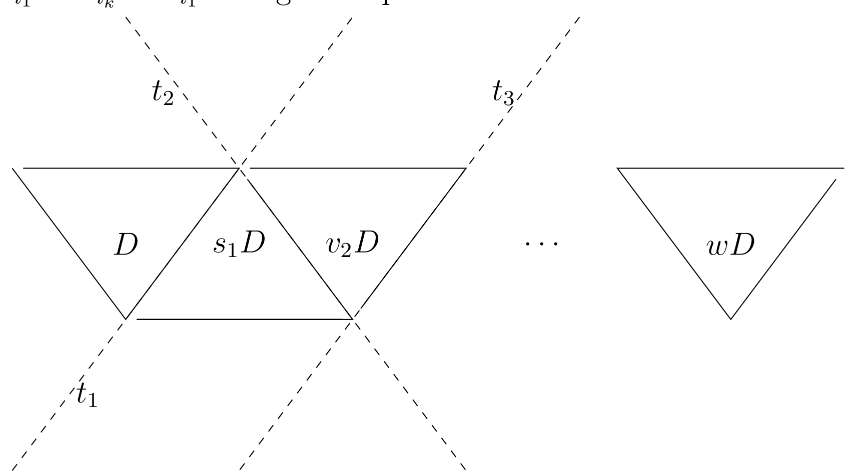
**September 29 – Inversions and length.** Last time, we defined $T = \{us_iu^{-1} | i = 1, \dots, n, u \in W\}$, the set of conjugates of simple generators. We saw the bijection:

$$\begin{array}{lcl} T & \longleftrightarrow & \{\beta^\perp \ : \ \beta \in \Phi\} \\ t & \longmapsto & \mathrm{Fix}(t) \\ ws_iw^{-1} & \longleftarrow & (w\alpha_i)^\perp. \end{array}$$

We introduce not standard, but useful notation; for $x \in V^\vee - \bigcup_{\beta\in\Phi} \beta^\perp$, set

$$I(x) = \{t \in T \ : \ \mathrm{Fix}(t) \text{ separates } D \text{ and } x\}.$$

If $x \in wD^\circ$, what is this? To get an idea, we let $w = s_{i_1}s_{i_2}\dots s_{i_\ell}$, $v_k = s_{i_1}\dots s_{i_k}$, and $t_k = v_kv_{k-1} = s_{i_1}\dots s_{i_k}\dots s_{i_1}$. We get the picture:




Walking from $D$ to $wD$, our reflections are the $t_i$s. In the above notation, we get another characterization of $I(x)$,

$$I(x) = \{t \in T | t \text{ occurs an odd number of times in } t_1, \dots, t_\ell\}.$$

This description is purely algebraic, while our other is geometric.

**Corollary.** The set $I(x)$ for $x \in wD^\circ$ depends only on $w$, not on the Cartan matrix.

We write $\mathrm{inv}(w)$ for $I(x)$. These are the **inversions** of $W$.

**Corollary.** For $t \in T$ and $w \in W$, the parity of the number of times $t$ occurs as $t_k$ is independent of the choice of word.

**Example.** In $A_2$, let $w_0 = s_1s_2s_1 = s_2s_1s_2$. With both forms, we get the same set of $t_i$s, just in a different order:

$$\begin{array}{rcc} & s_1s_2s_1 & s_2s_1s_2 \\ t_1 = & s_1 & s_2 \\ t_2 = & s_1s_2s_1 & s_2s_1s_2 = s_1s_2s_1 \\ t_3 = & s_1s_2s_1s_2s_1 = s_2 & s_2s_1s_2s_1s_2 = s_1. \end{array}$$

**Corollary.** If $x \in wD^\circ$, then $\#I(x) < \infty$. Similarly, if $x \in wD$, then $\{\beta \in \Phi^+/\mathbb{R}_{\geq 0} \ : \ \langle \beta, x\rangle < 0\}$ is finite.

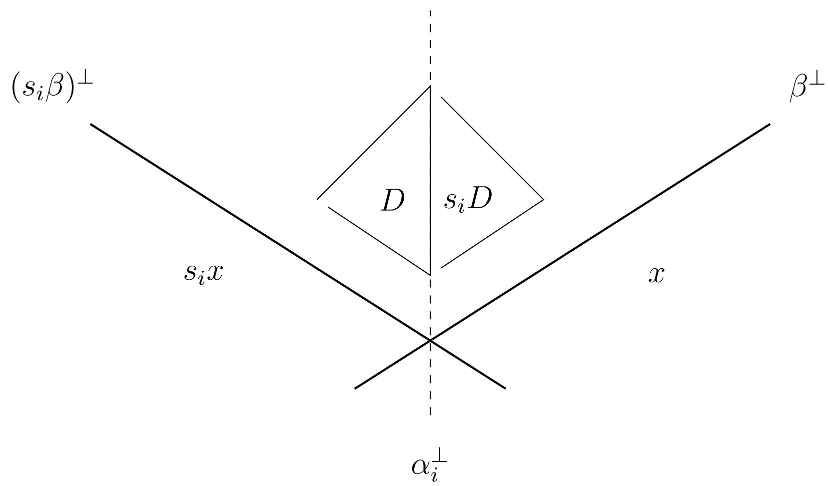
This corollary gives us a way to prove things aren't in the Tits cone. We now prove the converse result:

**Theorem.** Let $x \in V^\vee - \bigcup_{\beta\in\Phi}$, and $\#I(x) = \ell < \infty$. Then, $x \in wD^\circ$ for some $w$ with $\ell(w) = \ell$.

**Lemma.** For $x \in V^\vee - \bigcup_{\beta\in\Phi} \beta^\perp$,

- if $s_i \notin I(x)$, then $I(s_i x) = \{s_i\} \cup s_i I(x) s_i$, and
- if $s_i \in I(x)$, then $I(s_i x) = s_i I(x) s_i - \{s_i\}$.

*Proof.* If $t = s_i$, then $x$ is on one side of $\alpha_i^\perp$ and $s_i x$ is on the other, so $s_i \in I(x)$ if and only if $s_i \notin I(s_i x)$.




Now suppose that $t \neq s_i$. Let $t \in T$ reflect over $\beta^\perp \in \Phi^+$. The hyperplane $\beta^\perp$ separates $x$ from $D$ if and only if $s_i\beta^\perp$ separates $s_i x$ from $s_i D$. But $D$ and $s_i D$ are on the same side of $s_i\beta^\perp$. So $\beta^\perp$ separates $x$ from $D$ if and only if $s_i\beta^\perp$ separates $s_i x$ from $D$. We conclude that $t \in I(x)$ if and only if $s_i t s_i^{-1} \in I(s_i x)$.

□

We now prove the main theorem of the day.

*Proof.* We induct on $\ell$. If $\ell = 0$, then $I(x) = \varnothing$, and $x \in D^\circ$. We can take $x \in D^\circ$ and $w = 1$.

If $\ell > 0$ but finite, then $x \notin D^\circ$. So some $\alpha_i^\perp$ separates $x$ and $D^\circ$, giving us $s_i \in I(x)$. Then, we note $I(s_i x) = s_i I(x) s_i - \{s_i\}$. By induction, $s_i x \in w' D^\circ$ for some $w'$ with $\ell(w') = \ell - 1$ and $x \in s_i w' D^\circ$. Since $s_i \notin I(s_i x)$, this means $\langle \alpha_i, \_ \rangle > 0$ on $s_i x$, and $s_i x$ is a left ascent of $s_i w'$.

Therefore, $\ell(w) = \ell(w') + 1 = \ell$. □

**Corollary.** A word $s_{i_1} s_{i_2} \dots s_{i_\ell}$ is reduced if and only if the sequence $t_1, \dots, t_k$ has no repeats.

*Proof.* Let $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$. We know $\ell(w) = \#\{t_i \text{ occuring an odd number of times}\}$, so $\ell(w) = \ell$ if and only if each $t_i$ occurs exactly once. □

In practice, compute $\alpha_{i_1}$, $s_{i_1}\alpha_{i_2}$, $s_{i_1}s_{i_2}\alpha_{i_3}$, …, and $s_{i_1} \dots s_{i_{\ell-1}}\alpha_{i_\ell}$ to see if a word is reduced.

**Example.** In $D_4$, is $s_0s_1s_2s_3s_0$ reduced? Computing the list from above, we get:

$$\begin{aligned} &\alpha_0 \\ s_0\alpha_1 &= \alpha_0 + \alpha_1 \\ s_0s_1\alpha_2 &= \alpha_0 + \alpha_2 \\ s_0s_1s_2\alpha_3 &= \alpha_0 + \alpha_3 \\ s_0s_1s_2s_3\alpha_0 &= 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \end{aligned}$$

These are distinct, so $\ell(s_0s_1s_2s_3s_0) = 5$.

**Corollary.** The Tits cone, $\mathrm{Tits}(w)$, is the set $\{x \in V^\vee |$ finitely many $\mathrm{Fix}(t)$ separate $x$ and $D^\circ\}$.

**Corollary.** The Tits cone, $\mathrm{Tits}(w)$, is convex.

**Corollary.** If $|W| < \infty$, $\mathrm{Tits}(w) = V^\vee$, so the $wD^\circ$ are the connected components of

$$V^\vee - \bigcup_{\beta\in\Phi} \beta^\perp.$$

**Corollary.** If $|W| = \infty$, then $-D^\circ \cap \mathrm{Tits}(w) = \varnothing$

In the finite case, we define $w_0 \in W$ to be the element so that $w_0D^\circ = -D^\circ$. For this element, we have $\ell(w_0) = \#T$, and $\ell(w) < \#T$ for any $w \neq w_0$. For this reason, $w_0$ is the longest word.

**October 2 – David pays his debts.** We have two procedures that we want to verify are mutually inverse. Today's goal is to do exactly that, and answer the question from the end of September 11. We start by recording two "running" algorithms from past lectures.

Algorithm 1: Start with $W$ a finite orthogonal reflection group. Take

$$\Phi = \{\text{normals to hyperplanes}\}.$$

Choose $\rho$ not in any $\beta^\perp$; this splits $\Phi = \Phi^+ \sqcup \Phi^-$. Let $\alpha_1, \ldots, \alpha_k$ be simple roots in $\Phi^+$ with reflections $s_1, \ldots, s_k$. The $\alpha_i$ will be linearly independent, and the angle between $\alpha_i$ and $\alpha_j$ is $\pi(1 - 1/m_{ij})$ where $m_{ij}$ is the order of $s_is_j$. We get a positive-definite Cartan matrix $A$ with entries $A_{ij} = \alpha_i \cdot \alpha_j$.

Algorithm 2: Start with positive-definite Cartan matrix with entries $A_{ij}$. Find $\alpha_1, \ldots, \alpha_k \in V$ and $\alpha_1^\vee, \ldots, \alpha_k^\vee \in V^\vee$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$. Note that the $\alpha_j$'s will be linearly independent: indeed, if $\sum_i c_i\alpha_i = 0$, then $\langle \sum_i c_i\alpha_i^\vee, \sum_j c_j\alpha_j \rangle = 0$. However, the latter value can be expressed as $\langle \Sigma, \Sigma \rangle = c^TAc$ and $A$ is positive definite, yielding a contradiction.

We can define an inner product on $V$ (so $V \cong V^\vee$) and can think of $V = V^\vee$. The $s_i$, given by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$, generate some subgroup $W \subseteq \mathrm{GL}(V)$.

So Algorithm 1 associates to each finite orthogonal reflection group a positive-definite Cartan matrix, while Algorithm 2 associates to any positive-definite Cartan matrix an orthogonal reflection group.

**Claim 1:** The output of Algorithm 2 is **finite** when $A$ is positive definite.

*Proof of Claim 1.* Let $\Sigma$ be the unit sphere in $V$ and let $\Delta = \Sigma \cap D$. We equip $\Sigma$ with a Riemmannian metric by restricting the inner product from $V$, so we can talk about volumes of subsets of $V$. All of the $w\Delta$ have disjoint interiors in $\Sigma$. So

$$\mathrm{Vol}(\Sigma) \geq \sum_{w\in W} \mathrm{Vol}(w\Delta) = |W| \,\mathrm{Vol}(\Delta)$$

where the equality is because $W$ preserves the inner product on $V$ and hence preserves the metric on $\Sigma$. Thus

$$|W| \leq \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\Delta)}. \qquad \square$$

Now that we know $|W| < \infty$, we know $V = \bigcup wD$ so $\Sigma = \bigcup w\Delta$ and $|W| = \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\Delta)}$.

Suppose that we compose in the following order:

$$W_{in} \overset{Alg\ 1}{\longmapsto} A \overset{Alg\ 2}{\longmapsto} W_{out}.$$

That is, $W_{in}$ is sent to $A$ by Algorithm 1, and $A$ in turn is sent to $W_{out}$ by Algorithm 2.

**Claim 2:** Is $W_{in}$ isomorphic to $W_{out}$?

*Proof of Claim 2.* By definition, $W_{out}$ is generated by $s_1^{out}, \ldots, s_k^{out}$ obeying $(s_i^{out} s_j^{out})^{m_{ij}} = 1$. Meanwhile, the elements $s_1^{in}, \ldots, s_k^{in}$ in $W_{in}$ obey $(s_i^{in} s_j^{in})^{m_{ij}} = 1$. So we have a group homomorphism $W_{out} \to W_{in} \subseteq \mathrm{GL}(V)$.

We showed $W_{out} \to \mathrm{GL}(V)$ is injective, so $W_{out} \subseteq W_{in}$. The group $W_{in}$ is generated by reflections, so it is enough to check every reflection $t$ in $W_{in}$ is in the image of $W_{out}$. Suppose that $H = \mathrm{Fix}(t)$ and that $H = \beta^{\perp}$ for some $\beta \in \Phi_{in}$. If $H = \beta^{\perp}$ for some $\beta \in \Phi_{out}$, then $t$ is the orthogonal reflection over $\beta^{\perp}$ and $t \in W_{out}$. If not, $H$ passes through $wD^{\circ}$ for some $w \in W_{out}$. Up to replacing $t$ by $w^{-1}tw$, we may assume $H$ passes through $D^{\circ}$. Then

$$\beta \notin \mathrm{Span}_{\mathbb{R}^+}(\alpha_1, \ldots, \alpha_k) \cup -\mathrm{Span}_{\mathbb{R}^+}(\alpha_1, \ldots, \alpha_k).$$

But that means we failed to take the correct simples in Algorithm 1, a contradiction. $\square$

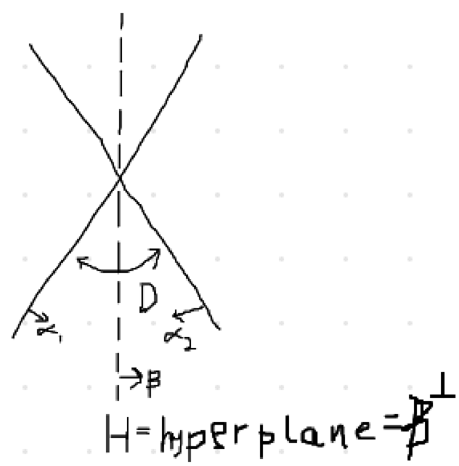



FIGURE 12. Proof of Claim 2.

Now we compose in the opposite order. Suppose we start with a positive-definite Cartan matrix $A_{in}$ and we construct

$$A_{in} \overset{Alg\ 2}{\longmapsto} W \overset{Alg\ 1}{\longmapsto} A_{out}.$$

Under Algorithm 2 we consider the following data: $\alpha_1^{in}, \ldots, \alpha_k^{in} \in V$ and a positive-definite inner product on $V$ with $\langle s_i^{in} \rangle = W$. Under Algorithm 1 we consider the following data: $\Phi = \Phi^+ \cup \Phi^-$, where $\alpha_1^{out}, \ldots, \alpha_k^{out} \in \Phi^+$ are the simples of $\Phi^+$. We want to verify that:

- $\langle \alpha_i^{out}, \alpha_j^{out} \rangle = \langle \alpha_i^{in}, \alpha_j^{in} \rangle$.
- If $\rho$ is such that $\langle \rho, \alpha_i^{in} \rangle > 0$, then $\{\alpha_i^{in}\} = \{\alpha_j^{out}\}$.

First of all, what if we choose $\rho$ such that $\langle \alpha_i^{in}, \rho \rangle > 0$?

From Problem Set 4, Problem 2, all reflections in $W$ are over some $(w\alpha_j^{in})^\perp$. Thus we conclude that $\Phi_{out} = \{w \cdot \alpha_j^{in}\}$. Now

$$\{w \cdot \alpha_j^{in}\} = \{\text{positive combinations of the } \alpha_j^{in}\text{'s}\} \sqcup \{\text{negative combinations of the } \alpha_j^{in}\text{'s}\}.$$

By our choice of $\rho$, the functional $\langle \rho, \cdot \rangle$ is positive on first set and negative on the second set.

Thus $\Phi^+$, as defined by $\langle \rho, \cdot \rangle$, will be positive combinations of the $\alpha_j^{in}$'s; the latter will be the simple roots.

What about some other $\rho$? That $\rho$ must lie in $wD^\circ$ for some $w \in W$, so $w^{-1}\rho \in D^\circ$. The positive roots for $\rho$ are $w \cdot \{\text{positive roots for } D^\circ\} = \{w \cdot \alpha_1^{in}, \dots, w \cdot \alpha_k^{in}\}$.

**Note to the reader:** There were a lot of topics which probably should have gotten covered at this point which were tossed onto problem sets instead. I wound up taking a day to cover them on October 20. If you are reading these notes without the class, you might want to go there now.

**October 4 - Crystallographic groups.**

**Main Idea.** Today, we started talking about when Coxeter groups preserve some lattice. We begin by recalling the following definition:

**Definition.** Let $V$ be a vector space. Then, $\Lambda \subseteq V$ is called a **lattice** iff it is a discrete additive subgroup of $V$ which spans $V$ as a vector space.

**Example.** $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ and $\mathbb{Z} \cdot \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \right\rangle \subseteq \mathbb{R}^2$ are both lattices, but $\mathbb{Z} \cdot \langle 1, \pi \rangle \subseteq \mathbb{R}$ is not (as it isn't discrete).

**Theorem.** Lattices are always of the form $\mathbb{Z} \cdot \langle w_1, \dots, w_n \rangle$ for $(w_1, \dots, w_n)$ a basis of $V$

**Definition.** For a lattice $\Lambda$ in $V$, the **dual lattice** $\Lambda^\vee \subseteq V^\vee$ is $\{x \in V^\vee : \langle x, \beta \rangle \in \mathbb{Z} \ \forall \beta \in \Lambda\}$

Note that $\Lambda_1 \subseteq \Lambda_2 \iff \Lambda_1^\vee \supseteq \Lambda_2^\vee$.

**Definition.** For $W$ acting on $V$, we say that $W$ **preserves** $\Lambda$ iff $w(\Lambda) \subseteq \Lambda$ for all $w \in W$.

Note that if $W$ preserves $\Lambda$, $W$ also preserves $\Lambda^\vee$ (acting via the dual action).

**Main Idea.** We will show that, very roughly, if $W$ is a Coxeter group, $W$ preserves some lattice in $V$ iff $A_{ij} \in \mathbb{Z}$. More specifically, we have the following two main theorems:

**Theorem** (Theorem 1)**.** If $W$ (a Coxeter group) preserves a lattice in $V$, then we can rescale that $\alpha_i$ (so $A_{ij} \rightsquigarrow \frac{c_i}{c_j} A_{ij}$) to make $A_{ij} \in \mathbb{Z}$. Furthermore, I can choose this rescaling such that $\mathbb{Z}\langle \alpha_i \rangle \subseteq \Lambda \subseteq (\mathbb{Z}\langle \alpha_i^\vee \rangle)^\vee$ (and dually, $\mathbb{Z}\langle \alpha_i^\vee \rangle \subseteq \Lambda^\vee \subseteq (\mathbb{Z}\langle \alpha_i \rangle)^\vee$

**Theorem** (Theorem 2)**.** If all $A_{ij} \in \mathbb{Z}$ and $\mathbb{Z}\langle \alpha_i \rangle$ and $\mathbb{Z}\langle \alpha_j^\vee \rangle$ are lattices, then $W$ preserves them and preserves any lattice $\Lambda$ between them.

Note that this requirement that these be lattices prevents this from being a full converse of Theorem 1.

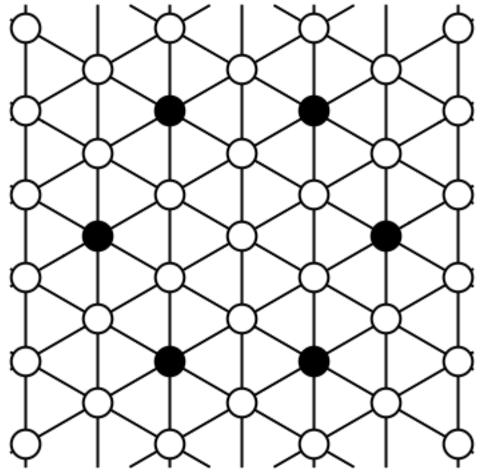


FIGURE 13. The $A_2$ roots and weight lattice

**Definition.** We define the following lattices:

**root lattice:** $\mathbb{Z}\langle\alpha_i\rangle$
**weight lattice:** $(\mathbb{Z}\langle\alpha^\vee\rangle)^\vee$
**coroot lattice:** $\mathbb{Z}\langle\alpha^\vee\rangle$
**coweight lattice:** $(\mathbb{Z}\langle\alpha\rangle)^\vee$

**Example.** Let $W = A_{n-1} \simeq S_n$, $V = (1,\dots,1)^\perp \subseteq \mathbb{R}^n$ and $V^\vee = V^\vee = \mathbb{R}^n/\mathbb{R}(1,\dots,1)$

We can then identify $V$ and $V^\vee$ via $V \hookrightarrow \mathbb{R}^n \twoheadrightarrow V^\vee$

Then, we can have $\alpha_i = e_{i+1} - e_i$ and $\alpha_i^\vee = e_{i+1} - e_i$

Our root lattice is then $\mathbb{Z}\langle\alpha_i\rangle = (1,\dots,1)^\perp \cap \mathbb{Z}^n$ and our weight lattice is $(\mathbb{Z}\langle\alpha^\vee\rangle)^\vee = \left\{(x_1,\dots,x_n) \in (1,\dots,1)^\perp : x_i - x_j \in \mathbb{Z} \forall i,j\right\} = \mathbb{Z}\langle\alpha_i\rangle + \mathbb{Z}\left(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n},\frac{1}{n}-1\right)$

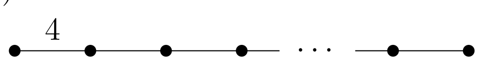
Thus, $(\mathbb{Z}\langle\alpha^\vee\rangle)^\vee/\mathbb{Z}\langle\alpha_i\rangle \simeq \mathbb{Z}/n\mathbb{Z}$ and so the lattices preserved by $W$ correspond to subgroups of $\mathbb{Z}/n\mathbb{Z}$.

**Definition.** A Cartan matrix is called **crystallographic** if all $A_{ij} \in \mathbb{Z}$.

A Coxeter group is called **crystallographic** if all $m_{ij} \in \{1,2,3,4,6,\infty\}$.

**Example.** In the following diagrams, the origin is at the center, the black circles are the generators of the root lattice, the lines are where pairing with some $\alpha_i^\vee$ gives an integer and the open circles are the weight lattice.

**Example.** We recall that, as groups $B_n = C_n$, however, they correspond to different crystallographic structures (for $n > 2$) as we will show here:

Recall that they correspond to 


Thus, we have that $A_{ii} = 2, A_{12}A_{21} = 2$, $A_{i,i+1}A_{i+1,i} = 1$ for $i > 1$ and $A_{ij} = 0$ if $|i-j| > 1$. Thus, for $A_{ij}$ to be integers there are two possibilities for the Cartan matrix:

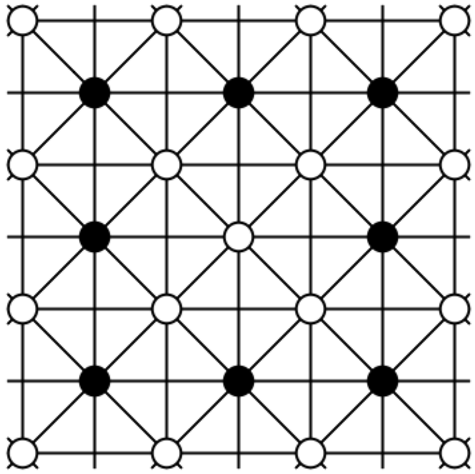


FIGURE 14. The $B_2 = C_2$ roots and weight lattice

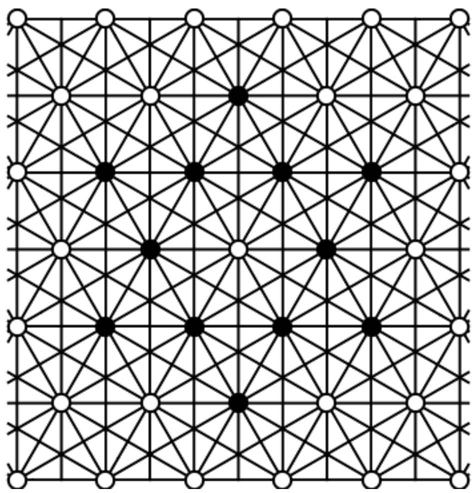


FIGURE 15. The $G_2$ roots and weight lattice

$$\text{Either} \begin{bmatrix} 2 & -2 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{bmatrix} \text{OR} \begin{bmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{bmatrix}$$

These give us the root systems
$B_n$: $\Phi = \{\pm e_k, \pm e_i \pm e_j\}, \Phi^\vee = \{\pm 2e_k, \pm e_i \pm e_j\}$, while $C_n$ has $\Phi$ and $\Phi^\vee$ reversed.

These look like $B_2$ : $\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot \end{array}$ and $C_2$ : $\begin{array}{ccccc} & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & 0 & & \cdot \\ & \cdot & & \cdot & \\ & & \cdot & & \end{array}$ so in $B_2$ and $C_2$ can be made to have the same crystallographic structure by a change in coordinates, but that isn't true for higher dimensions as most roots will be the longer length in $B_n$ and most will be the shorter length in $C_n$.

For $B_n$, we have $\mathbb{Z}\langle\alpha_i\rangle = \mathbb{Z}^n$, while $(\mathbb{Z}\langle\alpha_i^\vee\rangle)^\vee = \{(x_1,\ldots,x_n) : 2x_k, \pm x_i \pm x_j \in \mathbb{Z}\} = \left(\mathbb{Z}^n + \left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right) \cup \mathbb{Z}^n$.

For $C_n$, we have $\mathbb{Z}\langle\alpha_i\rangle = \{(a_1,\ldots,a_n) \in \mathbb{Z}^n : \sum a_i \equiv 0 (\text{mod } 2)\}$, while $(\mathbb{Z}\langle\alpha_i^\vee\rangle)^\vee = \mathbb{Z}^n$.

So while both have $(\mathbb{Z}\langle\alpha_i^\vee\rangle)^\vee \supsetneq \mathbb{Z}\langle\alpha_i\rangle$, in $B_n$, the smaller lattice has an orthonormal basis, while in $C_n$, it is the larger lattice.

**Theorem.** The finite crystallographic cases are:

| | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $(\mathbb{Z}\alpha_i^\vee)^\vee / \mathbb{Z}\alpha_i$ | $\mathbb{Z}/(n+1)\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \times \mathbb{Z}/3$ | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | 1 | 1 | 1 |

*Proof of Theorem 2.* Recall $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$ so if all $A_{ij} \in \mathbb{Z}$, then $s_i$ acts on $\mathbb{Z}\alpha_j$ by an integer matrix and $W$ preserves $\mathbb{Z}\alpha_j$. Likewise $W$ preserves $\left(\mathbb{Z}\alpha_j^\vee\right)^\vee$.

To show $W$ preserves any $\Lambda$ between these, it is enough to show that $W$ acts trivially on $(\mathbb{Z}\alpha_i^\vee)^\vee / \mathbb{Z}\alpha_i$.

Let $w \in (\mathbb{Z}\alpha_i^\vee)^\vee$. Then, $s_i(w) = w - \langle\alpha_i^\vee, w\rangle\alpha_i \equiv w (\text{mod } \mathbb{Z}\alpha_i)$. □

*Proof of Theorem 1.* Let $W$ preserve $\Lambda$. Then, we claim that $\mathbb{R}\alpha_i \cap \Lambda \neq (0)$.

To see this, note that $\Lambda$ spans $V$, so it contains $\lambda \notin (\alpha_i^\vee)^\perp$. Then, $s_i(\lambda) = \lambda + c\alpha_i$ for some $c \notin 0$, so $c\alpha_i \in \Lambda$. Thus, the claim holds.

Sine the lattice is discrete, $\mathbb{R}\alpha_i \cap \Lambda = \mathbb{Z}\cdot(c\alpha_i)$ for some $c \in \mathbb{R}_{\neq 0}$. Rescale so $c = 1$. Now, $\mathbb{Z}\langle\alpha_i\rangle_{i=1,\ldots,n} \subseteq \Lambda$.

Then, $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i \in \Lambda$, so $A_{ij}\alpha_i \in \Lambda$ and $A_{ij} \in \mathbb{Z}$.

Finally, for any $\lambda \in \Lambda$, $s_i(\lambda) = \lambda - \langle\alpha_i^\vee, \lambda\rangle\alpha_i \in \Lambda$ so $\langle\alpha_i^\vee, \lambda\rangle\alpha_i \in \Lambda$, which means that $\langle\alpha_i^\vee, \lambda\rangle \in \mathbb{Z}$. □

**October 6 – The root poset and the highest root.** We first work through a homework problem.

**Problem Set 3, Problem 3** Let $V^\vee = \{(a_i)_{i\in\mathbb{Z}} \in \mathbb{R}^\mathbb{Z} : a_{i+n} - a_i$ is independent of $1\}$. An element $f$ in the affine symmetric group $\tilde{A}_{n-1}$ acts on $V^\vee$ by $(f(a))_i = a_{f^{-1}(i)}$.

Choose a basis $\{e_1, e_2, ...e^n\}$ for $V^\vee$, then $(e_i)_j$=1 if $j \equiv i(n)$, and $(e_i)_j$=0 otherwise, and $f_j = j$. The reflection $s_i$ acts by an matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

The group looks like

$$\begin{bmatrix} S_n & * \\ 0 & 1 \end{bmatrix}$$

To work out $\alpha_i$ and $\alpha_i^\vee$, we need to find the form $1 - s_i$, that is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

So $\alpha_i^\vee = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and $\alpha_i = \begin{bmatrix} 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \end{bmatrix}$, $\alpha_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$.

The domain area is $D = \{x_1 e_1 + x_2 e_2 + \cdots + x_n e_n + yf : x_{i+1} + y \geq x_1, x_1 + y \geq x_n\}$, so we get $D = \{(a_i) : a_i \leq a_{i+1}\}$, and the Tits cone $Tits(W) = \{a_{i+n} - a_i > 0\}$.

Now go back to new materials.

**The root poset and highest root:** Let $(W, \Phi)$ be crystallographic, and $\alpha_i$ are linear independent, so $\Phi \subset \mathbb{Z}\{\alpha_i\}$, and $\Phi^+ \subset \mathbb{Z}_{\geq 0}\alpha_i$. We partially order $\Phi^+$ by the following rule: we say $\sum b_i \alpha_i \geq \sum c_i \alpha_i$ if $b_i \geq c_i$ for all $i$.

**Example.** Lets consider the root system $\Phi^+$ for $B_2$. On the left we draw the roots in black and the cover relations of the poset in blue:

**Proposition.** In finite type, the root poset is graded by the height $\mathrm{ht}(\sum b_i \alpha_i) = \sum b_i$.

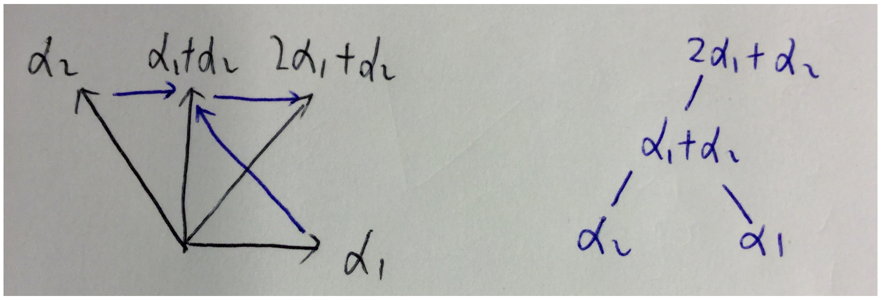



FIGURE 16. Order of roots of $B_2$

This was not proved in class. Thanks to Peter McNamara [1] for providing a short proof.

**Lemma.** Let $\Phi$ be a finite crystallographic root system and let $\alpha$ and $\beta \in \Phi$. If $\langle \alpha^\vee, \beta \rangle < 0$ and $\alpha \neq -\beta$, then $\alpha + \beta \in \Phi^+$. If $\langle \alpha^\vee, \beta \rangle > 0$ and $\alpha \neq \beta$ then $\beta - \alpha \in \Phi$ and, if we further assume $\alpha$ is a simple root, then $\beta - \alpha \in \Phi^+$.

*Proof.* We first consider the case that $\langle \alpha^\vee, \beta \rangle < 0$. We have $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \leq 3$ since our bilinear form is positive definite. The dot products $\langle \alpha^\vee, \beta \rangle$ and $\langle \beta^\vee, \alpha \rangle$ are integers with the same sign, and we have assumed that sign is negative. Thus, at least one of $\langle \alpha^\vee, \beta \rangle$ and $\langle \beta^\vee, \alpha \rangle$ is $-1$; without loss of generality say $\langle \alpha^\vee, \beta \rangle = -1$. Then the reflection of $\beta$ over $\alpha^\perp$ is $\beta - \langle \alpha^{-\vee}, \beta \rangle \alpha = \alpha + \beta$, so $\alpha + \beta \in \Phi$. As a sum of positive roots, it is clearly positive.

We first consider the case that $\langle \alpha^\vee, \beta \rangle > 0$. As before, we deduce that $\beta - \alpha \in \Phi$. If further more $\alpha$ is the simple root $\alpha_i$ and $\beta \neq \alpha_i$, then $\beta = \sum c_j \alpha_j$ and there is some $j \neq i$ for which $c_j > 0$. So $\beta - \alpha = \sum c_j \alpha_j - \alpha_i$ has positive coefficient on $\alpha_j$ and must be a positive root. ☐

*Proof that the root poset is graded by height.* It is obvious that, if $\beta_1 < \beta_2$ then $h(\beta_1) < h(\beta_2)$. What remains to be shown is that, if $\beta_1 < \beta_3$ and $h(\beta_3) - h(\beta_1) \geq 2$, then there is a $\beta_2$ with $\beta_1 < \beta_2 < \beta_3$. Let $\beta_3 = \beta_1 + \sum_{i \in I} c_i \alpha_i$ with $c_i > 0$. Let $\cdot$ be the positive definite inner product. Since $\sum_{i \in I} c_i \alpha_i \cdot \sum_{i \in I} c_i \alpha_i > 0$, there must be some $i$ for which $\alpha_i \cdot \sum_{i \in I} c_i \alpha_i > 0$. So $\alpha_i \cdot \beta_1 < \alpha_i \cdot \beta_3$. If $\alpha_i \cdot \beta_1 < 0$ then the lemma shows that $\beta_1 + \alpha_i \in \Phi^+$; set $\beta_2 = \beta_1 + \alpha_i$. If $\alpha_i \cdot \beta_3 > 0$, then the lemma shows that $\beta_3 - \alpha_i \in \Phi^+$; set $\beta_2 = \beta_3 - \alpha_i$. Either way, we have $\beta_1 < \beta_2 < \beta_3$. (We use that $h(\beta_3) - h(\beta_1) \geq 2$ to shows that these inequalities are strict. If $h(\beta_3) - h(\beta_1) = 1$, then the proof works until this detail, but we either get $\beta_1 = \beta_2$ or $\beta_2 = \beta_3$.) ☐

Obviously the minimal elements are $\alpha_i, \cdots \alpha_n$. When $\Gamma$ is connected, there is one maximal element, the ***highest root*** $\theta$. We will sketch a proof:

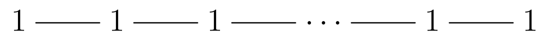
*Proof sketch.* From Problem Set 5, Problem 5, we know for each $x \in Tits(W)$, there is exactly one point in $Wx \cap D$.

Note that if $\theta$ is maximal in $\Phi$, then $\theta \in D$ after identifying $V$ with $V^\vee$, ie $V \cong V^\vee$. Indeed, if $\theta$ is not in $D$, then we can find an $\alpha_i$, such that $\theta \cdot \alpha_i < 0$, then $s_i \theta = \theta - \frac{2(\alpha_i, \theta)}{(\alpha_i, \alpha_i)} \alpha_i > \theta$. For each orbit in $\Phi$, is precisely one element in $D$, so in types $A$, $D$, $E$ we are done. In $B$, $C$, $F$, $G$, there is a long orbit and a short orbit. Let $\theta_{long}$ and $\theta_{short}$ be the unique root of each orbit in $D$. Using that $\Gamma$ is connected, we have $\gamma_1 \cdot \gamma_2 > 0$ for any $\gamma_1$, $\gamma_2$ nonzero vectors of $D$. So, in particular, $\theta_{short} \cdot \theta_{long} > 0$. By the above Lemma, $\pm(\theta_{long} - \theta_{short}) \in \Phi_+$. Assuming

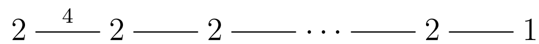
[1] https://mathoverflow.net/questions/283343

$\theta_{long} - \theta_{short} \in \Phi^+$ (in fact, this is always the correct sign), we see that $\theta_{long} > \theta_{short}$ in the root poset, so $\theta_{short}$ is not maximal. □

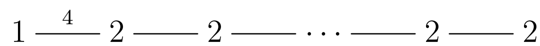
In $A_{n-1}$, highest root is $e_n - e_1 = \alpha_1 + \cdots \alpha_n$,



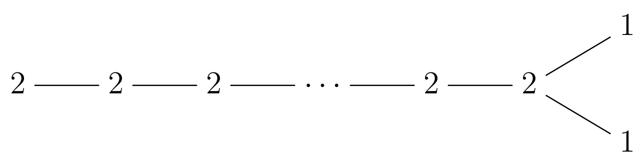

In $B_n$, the highest root is $e_{n-1} + e_n$.



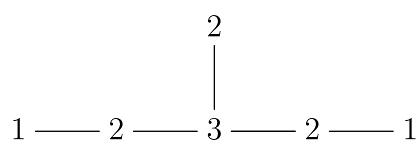

In $C_n$, the highest element is $2e_n$



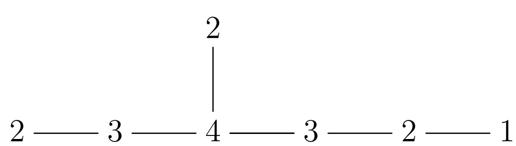

In $D_n$, the highest element is $e_{n-1} + e_n$



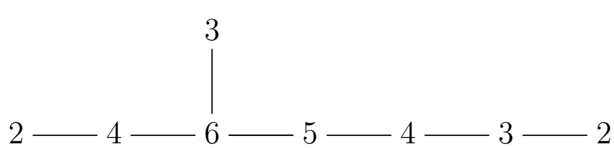

In $E_6$



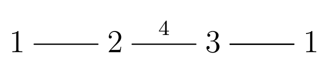

In $E_7$



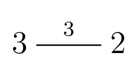

In $E_8$




In $F_4$




In $G_2$,



**October 9 - Affine groups.** Let $A$ be a symmetrizable Cartan matrix, means there exists some $d_i > 0$ with

$$d_i A_{ij} = d_j A_{ji}.$$

Set $B_{ij} = d_i A_{ij}$, then we could ask the following question:what will happen when $B$ is positive semi-definite? Here we get a positive semi-definite inner product on $V$, and roughly the answer to that question is

$$W looks like W_0 \ltimes (\mathbb{Z}\alpha_i^\vee),$$

where $W_0$ is finite Coxeter group, and $\mathbb{Z}\alpha_i^\vee$ is a lattice.

As usual, associated to the Coxeter group we could get a graph $\Gamma$, with edges $\Gamma_{ij}$ if $A_{ij} \neq 0$. We say $\Gamma$ is reduced if it is connected.

We first make the following claim:

**Claim:** The subspace $\ker(B)$ is at most 1 dimensional. And the vector in the kernel is a combination of $\alpha_i$'s with all same signs.

**Example.** When the Cartan matrix looks like

$$B = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & & -1 \\ -1 & & & -1 & 2 \end{pmatrix},$$

the kernel is

$$\ker(B) = \mathbb{R} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

*Proof of the Claim.* Let $\vec{c} = \sum c_i \alpha_i$ be in the $\ker(B)$, then $\sum c_i \alpha_i \cdot \sum c_i \alpha_i = 0$. We let $I_+ = \{i | c_i > 0\}$, $I_0 = \{i | c_i = 0\}$, and $I_- = \{i | c_i < 0\}$. So we could write $\vec{c} = \vec{c}_+ - \vec{c}_-$, where $\vec{c}_+, \vec{c}_-$ have $\geq 0$ entries with disjoint support. Then we have

$$\vec{c}_+ = \sum_{i \in I_+} c_i \alpha_i, \vec{c}_- = -\sum_{i \in I_-} c_i \alpha_i.$$

From $(\vec{c}_+ - \vec{c}_-) \cdot (\vec{c}_+ - \vec{c}_-) = 0$, we get

$$\vec{c}_+ \cdot \vec{c}_+ + \vec{c}_- \cdot \vec{c}_- = 2\vec{c}_+ \cdot \vec{C}_-,$$

where the left side is $\geq 0$ by the positive-semi-definiteness, and the right side is $\leq 0$ since $A_{ij} \leq 0$ for $i \neq j$. In this way, the condition we have makes everything become 0, and $\vec{c}_+, \vec{c}_-$ are null vectors of length 0, such that $I_+$ does not border $I_-$ is $\Gamma$.

Now we make the following claim:

**Claim** One of $I_+$ and $I_-$ is all of $\{1, 2, \ldots, n\}$, and the other is $\emptyset$, (here we already assumed that $\Gamma$ is connected). If not, let $i_0 \in I_0$ border $I_+$. Then we get

$$0 \leq (\vec{c}_+ + \delta\alpha_{i_0}) \cdot (\vec{c}_+ + \delta\alpha_{i_0}) = \vec{c}_+ \cdot \vec{c}_+ + 2\delta\vec{c}_+ \cdot \alpha_{i_0} + \delta^2 \alpha_{i_0} \cdot \alpha_{i_0},$$

where $\delta > 0$ is small enough, and the right side is small than 0, a contradiction.

Now we have shown that for $\vec{c}$ in kernel, all $c_i$ have the same sign. This actually forces 1-dimensionality: If $\vec{c}$ and $\vec{d}$ are non proportional, then for some scalars $p$ and $q$ and some coordinates $i$ and $j$, we have

$$pc_i + qd_i < 0, pc_j + qd_j > 0,$$

a contradiction.

□

Now let $\vec{c}$ be in $\ker(B)$, Then we have

$$\langle \alpha_j^\vee, \sum c_i\alpha_i\rangle = 0, \forall j.$$

Then we will have $\sum c_i\alpha_i \neq 0$. Otherwise if $\sum c_i\alpha_i = 0$, then $\sum c_i\langle x, \alpha_i\rangle = 0$ for all $x \in V^\vee$, which leads to $D^\circ = \varnothing$. And we denote by $\delta$ to be $\sum c_i\alpha_i$, then all $\alpha_i^\vee \in \delta^\perp$.

**Example.** Recall in recent homework, $\alpha$'s looked like

$$\alpha_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \\ \cdots \\ 1 \end{pmatrix}, \alpha_i^\vee = [0, \cdots, 1, -1, \cdot, 0].$$

We put $V_1^\vee = \{x \in V^\vee | \langle x, \delta\rangle = 1\}$, then $W$ preserves $V_1^\vee$. (This is because

$$\begin{aligned}\langle \delta, s_i x\rangle &= \langle \delta, -\langle \alpha_i, x\rangle\alpha_i^\vee\rangle \\ &= \langle \delta, x\rangle - \langle \alpha_i, x\rangle\langle \delta, \alpha_i^\vee\rangle,\end{aligned}$$

where the latter is 0.

We then descend the $\cdot$ from $V$ to $V/\mathbb{R}\delta$, and it becomes positive definite on $V/\mathbb{R}\delta$. Canonically $(V/\mathbb{R}\delta)^\vee \cong \delta^\perp \subset V^\vee$. So $\delta^\perp$ has positive definite inner product. And $V_1^\vee$ has a canonical matric:

$$d(x, y) := (x - y) \cdot (x - y),$$

which is preserved by $W$. That also means $W$ acts on $V_1^\vee$ through the Euclidean symmetry group

$$\mathcal{O}(\delta^\perp) \ltimes \delta^\perp.$$

Note that, on $V_1^\vee$, $\langle -, \delta\rangle = 1$, so $\beta^\perp \cap V_1^\vee$ and $(\beta + \delta)^\perp \cap V_1^\vee$ are parallel hyperplanes.

We will be proving

**Proposition.** We have

$$W = W_0 \ltimes \mathbb{Z}\Phi_0^\vee,$$

for some finite crystallographic $(W_0, \Phi_0^\vee)$.

**Example.** See the lattice of $\widetilde{A}_2$ in Figure 17.

Since $\delta = \sum c_i\alpha_i$ has $c_i > 0$, the linear form $\langle -, \delta\rangle$ is positive on $D\backslash\{0\}$. Set $D_1 = D \cap V_1^\vee$, then $D_1$ is a simplex, see Figure 18. (When $\Gamma$ is disconnected, the corresponding story will involve a product of simplices.)

Now we make the following claim:

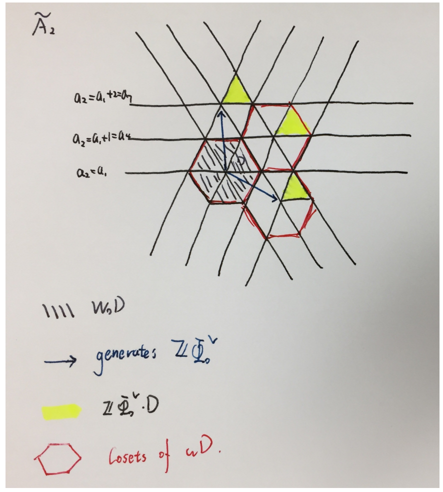



FIGURE 17. The geometry of $\widetilde{A}_2$

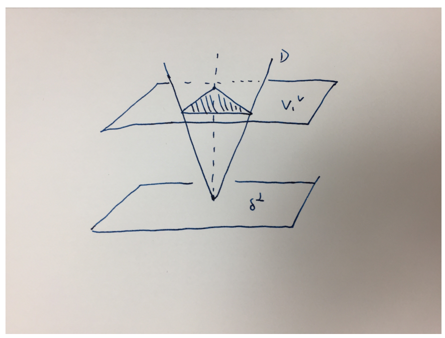



FIGURE 18. $D_1$ and $V_1^\vee$

**Proposition.** We have

$$\bigcup_{w\in W} wD_1 = V_1^\vee.$$

*Proof Sketch.* Since $D_1 \subset V_1^\vee$, and $W$ preserves $V_1^\vee$, so all $wD_1 \subset V_1^\vee$. If they did not fill, let $z$ be on the boundary of $\bigcup_{w\in W} wD_1$. Then $D_1$'s accumulate at $z$. But all $D_1$'s have same area and shape, which means they cannot accumulate at any point in $V_1^\vee$. □

At last, we claim the following:

**Proposition.** The image of $\Phi$ in $V/\mathbb{R}\delta$ is finite. Correspondingly, there are only finitely many directions of hyperplanes.

*Proof.* It is enough to show a lower bound for angle between images of $\beta_1^\perp$ and $\beta_2^\perp$ in $V/\mathbb{R}\delta$. If $\beta_1^\perp$ and $\beta_2^\perp$ in $V_1^\vee$ are not parallel, let $x \in \beta_1^\perp \cap \beta_2^\perp \cap V_1^\vee$. Then $w^{-1}\beta_1^\perp$ and $w^{-1}\beta_2^\perp$ meet on $D_1^\vee$ only at finitely many hyperplanes touching $D_1$. So $W$ acts on $V/\mathbb{R}\delta$ via finite reflection group $W_0$. □

**October 11 – Affine groups II. Summary from Last Class** If $A$ is a positive, semi-definite Cartan matrix, then we get a positive semi-definite inner product on $V$. Reduce to the case where $\Gamma$ is connected. Then the inner product $\cdot$ has signature $(++\ldots+0)$. Let

$\delta = \ker(\cdot)$. $W$ preserves hyperplanes in $V^\vee$ perpendicular to $\delta$. (See figure 18 from the notes from last class). Recall $V_1^\vee = \{x \in V^\vee : \langle \delta, x \rangle = 1\}$ where $V_1^\vee$ has a natural metric and $W$ acts on $V_1^\vee$ by maps in $\mathcal{O}(\delta^\perp) \ltimes \delta^\perp$ where $\mathcal{O}(\delta^\perp)$ corresponds to rotations and reflections while $\delta^\perp$ corresponds to translations.

We further showed that $D_1^\vee = D \cap V_1^\vee$ is a simplex, $\bigcup_{w \in W} wD_1^\vee$ fills $V_1^\vee$, and, when $\beta \in \Phi$, then $\beta^\perp \cap V_1^\vee$ is an affine hyperplane and $(\beta + \delta)^\perp$ is parallel to $\beta^\perp$. Since the image of $\Phi$ in $V/\mathbb{R}\delta$ is finite, there are only finitely many parallel classes of hyperplanes.

We have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P & \longrightarrow & W & \longrightarrow & W_0 & \longrightarrow & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \longrightarrow & \delta^\perp & \longrightarrow & \mathcal{O}(\delta^\perp) \ltimes \delta^\perp & \longrightarrow & \mathcal{O}(\delta^\perp) & \longrightarrow & 1 \end{array}$$

Here $P = W \cap \delta^\perp$. $W_0$ is a reflection group acting on $\delta^\perp$ and its dual preserves the image of $\Phi$ in $V/\mathbb{R}\delta$. So, since this image is finite, $W_0$ is finite.

**Claim 1:** $P$ is a lattice. Since the $P$-translates of $D_1^\vee$ are disjoint, $P$ is discrete.

*Proof.* For each $w \in W_0$, choose a lift $\tilde{w}$ to $W$ and set $Q = \bigcup \tilde{w} D_1^\vee$. Then $Q$ is bounded and $\bigcup_{p \in P} pQ$ fills $V_1^\vee$ so $\mathrm{rank}(P) = \dim(V_1^\vee)$. So $W_0$ is crystallographic. □

Recall: If $\beta_2 = \beta_a + k\delta$, then $\beta_2^\perp$ and $\beta_1^\perp$ are parallel. For any $\beta \in \Phi$, some $p \in P$ does not take $\beta^\perp$ to itself. So, there is $\beta + k\delta \in \Phi$ for some scalar $k$. Given $\beta \in \Phi$ we have $(\beta + \mathbb{R}\delta) \cap \Phi = \{\beta + \mathbb{Z}(k\delta)\}$ for some $k > 0$. Rescale so all $k$ are equal to 1. (Note $\Phi$ is still $W$-invariant because there were no choices made in that scaling.)

**Claim 2:** All $A_{ij}$ are integers. This was done badly in class; it will be fixed in the next class.

**Claim 3:** The short exact sequence $1 \to P \to W \to W_0 \to 1$ is semi-direct.

*Proof.* As seen in the commutative diagram above, the sequence $1 \to \delta^\perp \to \mathcal{O}(\delta^\perp) \ltimes \delta^\perp \to \mathcal{O}(\delta^\perp) \to 1$ is semi-direct. We would like to use a lift to show that the sequence in the top line of the diagram is semi-direct as well. Let $s_1^0, ..., s_{n-1}^0$ be simple generators for $W_0$. They are reflections over $n-1$ normals to linearly independent vectors. Lift each $s_i^0$ to $s_i \in W$. These are now reflections over $n-1$ transverse hyperplanes. They meet at some $x \in V_1^\vee$, a vertex of our hyperplane arrangement. The $s_i$ also satisfy $(s_i s_j)^{m_{ij}} = 1$ so $s_i^0 \to s_i$ extends to $W_0 \to W$. This means that we do have a splitting for the sequence $1 \to P \to W \to W_0 \to 1$. □

We can translate $D$ such that $\mathrm{Fix}(s_i), ..., \mathrm{Fix}(s_{n-1})$ are $n-1$ of the boundary walls of $D_1^\vee$. There is one more boundary wall.

**Claim:** We can write $V = V_0 \oplus \mathbb{R}$ where $W_0$ acts on $V_0$ and $\Phi = \mathbb{Z} + \Phi_0$ and the simple roots are $(\alpha_1^0, 0), ..., (\alpha_{n-1}^0, 0), (-\theta, 1)$ where $\theta$ is the highest root.

**October 13 – Affine Groups III.** Let's recall where we were. We split the map $W \twoheadrightarrow W_0$. This gave a copy of the root system $\Phi_0$ inside $\Phi$. We normalized the roots such that, for $\beta_0 \in \Phi_0$, we have $\beta_0 + \mathbb{R}\delta \cap \Phi = \beta_0 + \mathbb{Z}\delta$. So we have a splitting of $V$ as $V_0 \oplus \mathbb{R}\delta$ and a splitting of $V^\vee$ as $\delta^\perp \oplus V_0^\perp$. The $(n-1)$-dimensional vector spaces $V_0$ and $\delta^\perp$ have canonical inner products which identify them with each other. We identify $V_1^\vee$ with $\delta^\perp$, identifying the origin with $V_0^\perp \cap V_1^\vee$, where the $n-1$ lifted hyperplanes we described yesterday meet.

For $\beta_0 \in \Phi_0$, lets see how the reflection over $(\beta_0 + k\delta)^\perp$ acts on $V_1^\vee$. It acts by

$$x \mapsto x - \langle \beta_0 + k\delta, x\rangle \beta_0^\vee = x - (\langle \beta_0, x\rangle + k)\beta_0^\vee.$$

In other words, $x \mapsto t(x) + k\beta_0^\vee$ where $t$ is the reflection over $\beta_0^\perp$. We thus see that $W$ acts on $V_1^\vee$ by maps in $W_0 \ltimes \mathbb{Z}(\alpha_i^\vee)$.

Conversely, let's see that every map in $W_0 \ltimes \mathbb{Z}(\alpha_i^\vee)$ is in $W$. Clearly, $W_0 \subset W$. Suppose that we reflect over $\alpha_i^\perp$, and then over $(\alpha_i + \delta)^\perp$. The first map takes $x$ to $x - \langle \alpha_i, x\rangle \alpha_i^\vee$ and the second map takes us to

$$(x - \langle \alpha_i, x\rangle \alpha_i^\vee) - \left\langle x - \langle \alpha_i, x\rangle \alpha_i^\vee, \alpha_i + \delta \right\rangle \alpha_i^\vee = x - \langle \alpha_i, x\rangle \alpha_i^\vee - \langle \alpha_i, x\rangle \alpha_i^\vee + 2\langle \alpha_i, x\rangle \alpha_i^\vee + \alpha_i^\vee = x + \alpha_i^\vee.$$

so we have the translations to generate $\mathbb{Z}\alpha_i^\vee$.

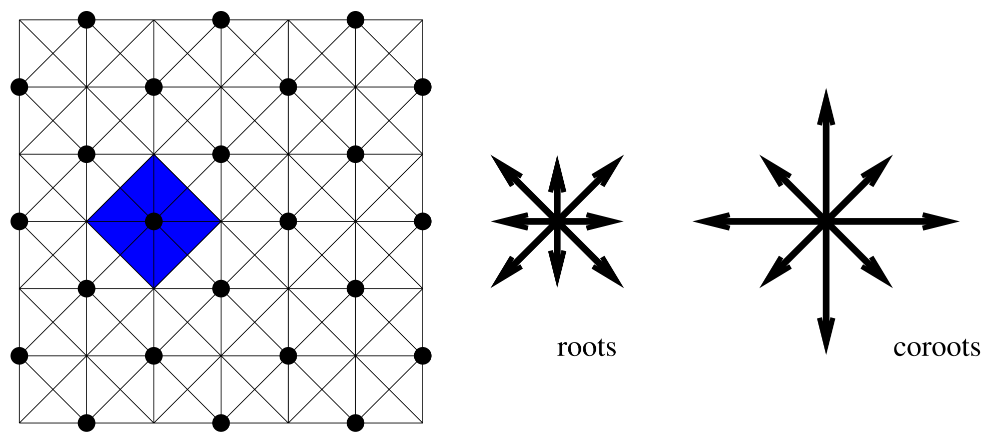
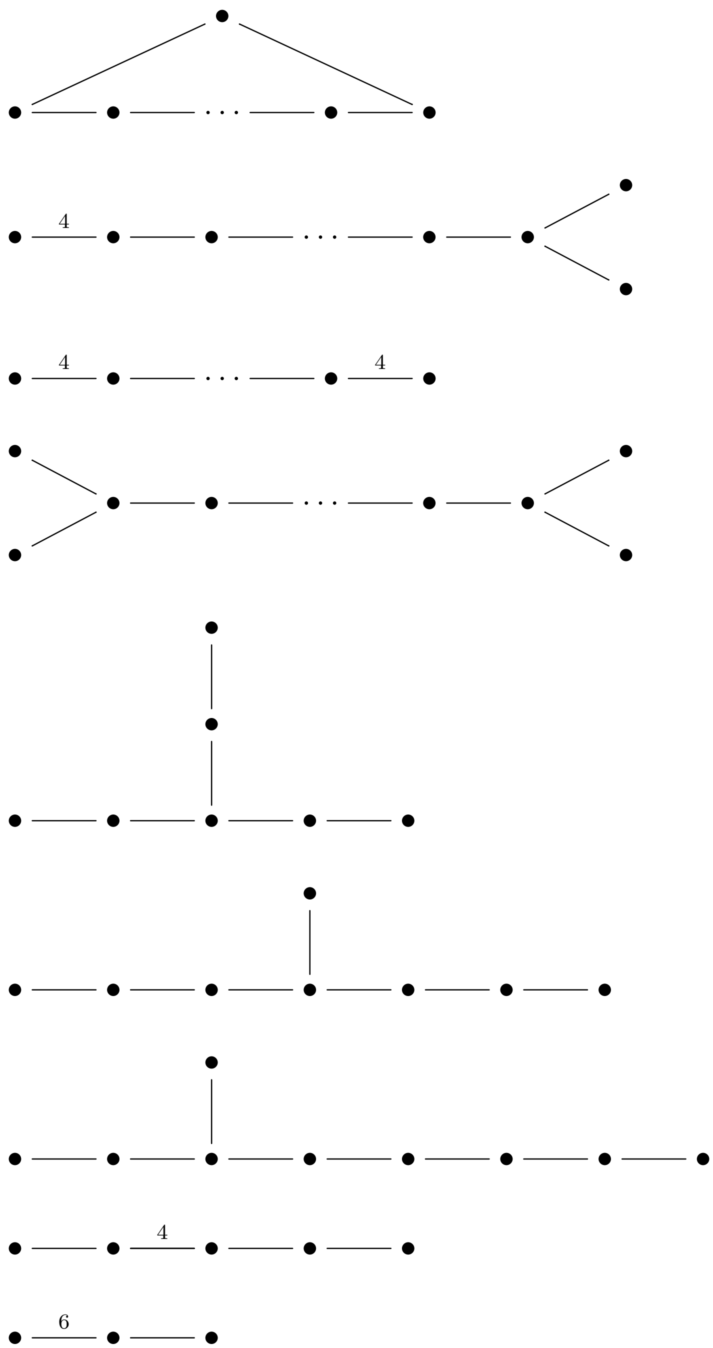



FIGURE 19. The $\tilde{B}_2$ arrangment

**Example.** Figure 19 shows the affine group $\tilde{B}_2$. The affine hyperplanes in the drawing are $\{x : \langle \beta_0, x\rangle = k\}$ for $\beta_0$ a root and $k \in \mathbb{Z}$. The solid black dots show the coroot lattice. The domains $\bigcup_{w \in W_0} wD$ are shaded in blue. Note that translations by the root lattice tile the plane with copies of the blue region, corresponding to the decomposition $W = W_0 \ltimes \mathbb{Z}(\alpha_i^\vee)$.

If $\alpha_1^\circ$, $\alpha_2^\circ$, $\dots$, $\alpha_{n-1}^\circ$ are the simple roots for $W_0$, then the simple roots for $W$ are $\alpha_1^\circ$, $\alpha_2^\circ$, $\dots$, $\alpha_{n-1}^\circ$ and $-\theta_{\text{high}} + \delta$ where $\theta_{\text{high}}$ is the highest root.

We take the opportunity to give the Coxeter diagrams of the affine groups. (This wasn't done in class.) Note that each diagram has one more dot than its subscript.




$\tilde{A}_n$

$\tilde{B}_n$

$\tilde{C}_n$

$\tilde{D}_n$

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$

$\tilde{F}_4$

$\tilde{G}_2$

**A fun identity:** Let $W_0$ be a finite irreducible crystallographic Coxeter group of rank $n-1$. Let $\theta$ be the highest root and write $\theta = \sum_{i=1}^{n-1} c_i \alpha_i$. Let $f$ be the index $(\mathbb{Z}\alpha_i)^\vee / \mathbb{Z}(\alpha_i^\vee)$. Then

$$|W_0| = (n-1)! c_1 c_2 \cdots c_{n-1} f.$$

(I've chosen my indexing to match the rest of the lecture. Sitting by itself, the result would look prettier if we called the group $W$ and the rank $n$, so we wrote $|W| = n! c_1 c_2 \cdots c_n f$.)

We consider fundamental domains for the lattices $(\mathbb{Z}\alpha_i)^\vee$ and $\mathbb{Z}(\alpha_i^\vee)$. Set

$$\Pi = \{x : 0 \le \langle x, \alpha_i \rangle \le 1,\ 1 \le i \le n-1\}.$$

$$\Delta = \bigcup_{w \in W_0} wD_1$$

So $\Pi$ is a fundamental domain for $(\mathbb{Z}\alpha_i)^\vee$ and $\Delta$ for $\mathbb{Z}(\alpha_i^\vee)$. In Figure 20, $\Delta$ is drawn in blue and $\Pi$ in yellow. We have translated $\Pi$ to make the figure more readable; drawn accurately, one of the acute vertices of $\Pi$ is at the center of $\Delta$.

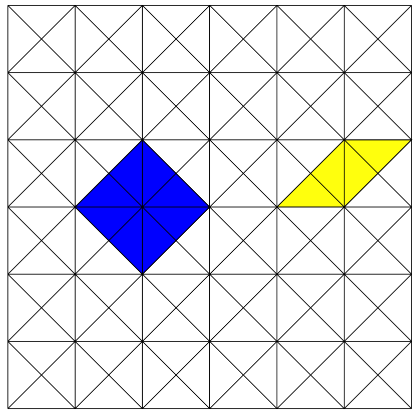


FIGURE 20. The domains $\Pi$ and $\Delta$ for $\tilde{B}_2$

So $\frac{\mathrm{Vol}(\Delta)}{\mathrm{Vol}(\Pi)} = f$. We have defined $\Delta$ to be covered by $|W_0|$ copies of $D_1^\vee$. We now want to count how many copies of $D_1$ are in $\Pi$. Let $\omega_1, \ldots, \omega_{n-1}$ be the dual basis to $\alpha_1, \ldots, \alpha_{n-1}$. So $\Pi = \{\sum x_i \omega_i : 0 \leq x_1, \ldots, x_{n-1} \leq 1\}$. The simplex $D_1$ has one vertex at 0 and the other vertices at $c_i^{-1}\omega_i$ (for $1 \leq i \leq n-1$). So

$$\mathrm{Vol}(D_1) = \frac{1}{(n-1)!} \frac{1}{c_1 c_2 \cdots c_{n-1}} \mathrm{Vol}(\Pi).$$

Putting it all together,

$$f = \frac{\mathrm{Vol}(\Delta)}{\mathrm{Vol}(\Pi)} = \frac{|W_0| \mathrm{Vol}(D_1)}{(n-1)! c_1 c_2 \cdots c_{n-1} \mathrm{Vol}(D_1)}$$

and we conclude

$$|W_0| = (n-1)! c_1 c_2 \cdots c_{n-1} f.$$

**October 18 – Hyperbolic groups.** Here is a summary of the different types of Coxeter groups that give pretty pictures:

- Positive definite bilinear forms correspond to finite Coxeter groups which look like triangulations of $S^{n-1}$.
- Positive semi-definite bilinear forms correspond to affine Coxeter groups which look like triangulations of $\mathbb{R}^{n-1}$.
- Bilinear forms with signature $(++\ldots+-)$ correspond to hyperbolic Coxeter groups which look like triangulation of $\mathbb{H}^{n-1}$.

**Hyperbolic geometry:** Let $V$ be an $n$-dimensional real vector space with a symmetric bilinear form, the dot product $\cdot$ with signature $(+^{n-1}-)$. Let $Q = \{\vec{v} \in V : \vec{v} \cdot \vec{v} = -1\}$

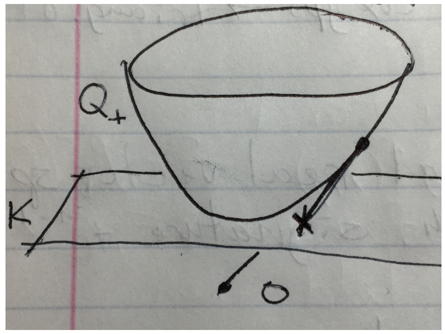



FIGURE 22. The point $x$ represents the image of a point in the Klein model

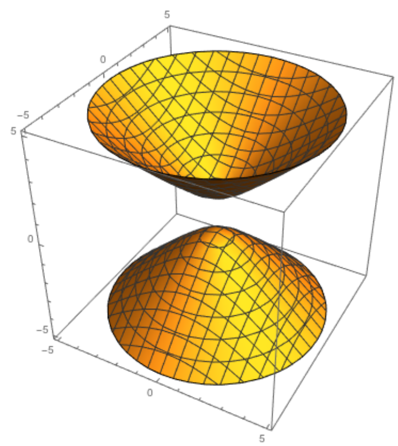


FIGURE 21. Hyperboloid $Q = Q_+ \cup Q_-$

E.g. $x^2 + y^2 - z^2 = -1$, see Figure 21. This is a hyperboloid of two sheets. Call this hyperboloid $Q$ and denote the sheets with positive and negative $z$ values by $Q_+$ and $Q_-$ respectively. Then we have the following isomorphism of manifolds: $Q_+ \cong \mathbb{R}^{n-1} \cong B(1) \subset \mathbb{R}^{n-1}$ where $B(1)$ denotes the ball of radius 1. The first isomorphism can be demonstrated with the map $\mathbb{R}^{n-1} \to Q_+$ defined by $(x, y) \mapsto (x, y, \sqrt{1 + x^2 + y^2})$. The isomorphism to $B(1)$ will be discussed later.

For $\vec{v} \in Q_+$, we get the tangent $T_{\vec{v}}Q_+ = \{\vec{x} \in V : \vec{v} \cdot \vec{x} = 0\} = \vec{v}^\perp$. Restricting $\cdot$ to $T_{\vec{v}}Q$ gives a dot product on $T_{\vec{v}}Q$ which is positive definite (because $V = \mathbb{R}\vec{v} \oplus (\vec{v})^\perp$). So $Q_+$ is naturally a Riemannian manifold, commonly called the hyperbolic place/space/disk.

Given $\vec{v} \in V$ with $\vec{v} \cdot \vec{v} > 0$, we can reflect over $\vec{v}^\perp$ in the following way:

$$\vec{x} \mapsto \vec{x} - 2\frac{\vec{v} \cdot \vec{x}}{\vec{v} \cdot \vec{v}}\vec{v}$$

This takes $Q_+$ to itself. Reflect over $\vec{v}^\perp \cap Q_+$. These are hyperplanes in $Q_+$.

Given $n = 3$ and $\dim Q_+ = 2$ there are two ways to draw this in the disk.

**Klein model** Take an affine plane, $K$, separating the origin from $Q_+$. Plot $\vec{x} \in Q_+$ at the intersection of $\mathbb{R}\vec{x}$ with $K$. See Figure 22. The picture will be contained within an open ball and hyperplanes are represented as affine hyperplanes in the open ball.

**Poincare model** Choose $\vec{v} \in Q_+$. Plot $\vec{x} \in Q_+$ at the intersection of the line through $-\vec{v}$ and $\vec{x}$ with $T_{\vec{v}}Q_+$. See Figure 23. In the Poincare model, lines are arcs of circles

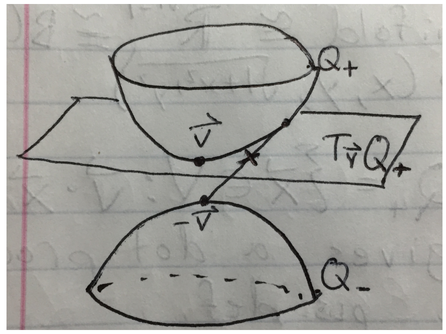



FIGURE 23. The point $x$ represents the image of a point in the Poincare model

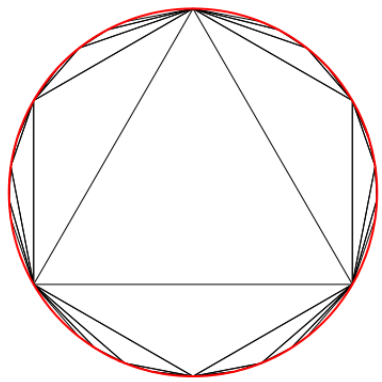


FIGURE 24. Triangulation in Klein model

perpendicular to the boundary of the disk. The Poincare model is conformal, meaning that angles are correct. It also puts things nearer to the center of the disk, so the diagram can render more detail.

**Example.** To compare the Klein and Poincare models, we will consider Cartan matrix and reflection group given as follows:

$$A = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \qquad W = \langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = 1 \rangle$$

The triangulation created in this example can be seen in the Klein model (Figure 24) or the Poincare model (Figure 25).

If we have $\alpha_1, ..., \alpha_n \in V$ with $\alpha_i \cdot \alpha_i = 2$ and $\alpha_i \cdot \alpha_j \in \{-2\cos\frac{\pi}{m} : m \geq 2\} \cup (-\infty, -2]$ and $\{x : \alpha_i \cdot x > 0\} \neq \emptyset$ we'll get a Coxeter group acting on $Q_+$. We can do this if $A$ is symmetric with signature $(+ + ... + -)$. Take $V$ to be the vector space on basis $\{\alpha_i\}$ and define $\cdot$ by $\alpha_i \cdot \alpha_j = A_{ij}$.

**Example.** If we want $m_{12} = m_{13} = m_{23} = 4$ (i.e. triangles with all angles equal to $\frac{\pi}{4}$) then use the Cartan matrix

$$\begin{bmatrix} 2 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 2 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 \end{bmatrix}$$

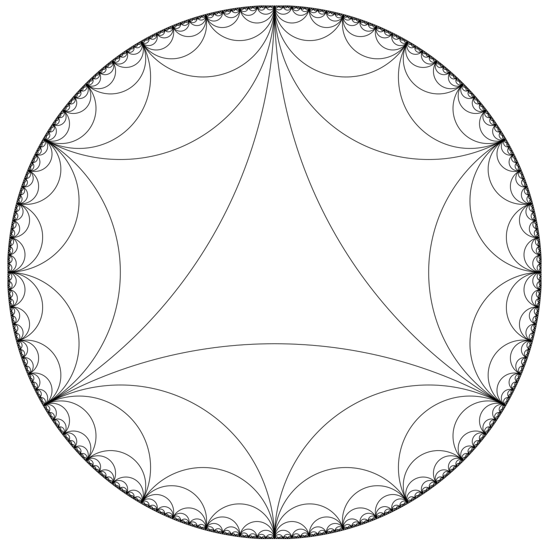


FIGURE 25. Triangulation in Poincare Model

Slightly more generally, if $\mathrm{sig}(A) = +^{k-1}0^{n-k}-$ then we can get $\alpha_1, ..., \alpha_n$ in $V$ where $\dim V = k$ and $\alpha_i \cdot \alpha_j = A_{ij}$ and we can hope to have $D \neq \emptyset$.

**Example.** Suppose we want to tile the disk with pentagons whose angles are all $\frac{\pi}{2}$. Then we need $m_{12} = m_{23} = ... = m_{51} = 2$ and $m_{13} = m_{24} = ... = \infty$. Then we get the signature $(++00-)$ and the Cartan matrix

$$A = \begin{bmatrix} 2 & 0 & \alpha & \alpha & 0 \\ 0 & 2 & 0 & \alpha & \alpha \\ \alpha & 0 & 2 & 0 & \alpha \\ \alpha & \alpha & 0 & 2 & 0 \\ 0 & \alpha & \alpha & 0 & 2 \end{bmatrix} \text{ where } \alpha \leq -2$$

Note: The diagram shown in class has $\alpha = -1 - \sqrt{5}$. If the $\alpha$ values were chosen to not all be equal, then we would get different side ratios on the pentagons.

How will $D$ and $\bigcup wD = \mathrm{Tits}(w)$ relate to $Q_+$? In all our examples so far, $\cdot$ is $\leq 0$ on $D$ and $\mathrm{Tits}(w)$ filled up one of the two components of $\{x : x \cdot x \leq 0\}$.

If $\mathrm{sig}(A) = (+^{n-1}-)$, then $D$ lies in $Q_+$ if every principal $(n-1) \times (n-1)$ minor of $A$ is positive definite and $D$ lies in $\overline{Q_+}$ if these minors are positive semi-definite.

**Example.**

$$A = \begin{bmatrix} 2 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 2 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 \end{bmatrix} \quad \text{Here, the minors } \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} \text{ are positive definite}$$

In this case, $\bigcup_{w \in W} wD$ fills $Q_+$.

**Example.**

$$A = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \quad \text{Here, we have } Q_+ \subset \bigcup_{w \in W} wD \subset \overline{Q_+}$$

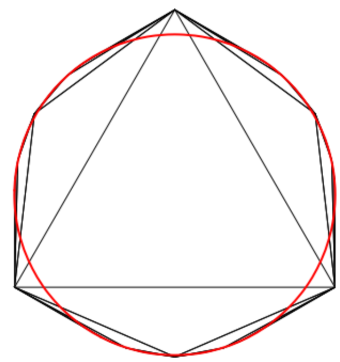


FIGURE 26. Klein model for an example with negative signature

**Example.** Finally, consider an example in which we allow negative signature. This type of example cannot be illustrated with the Poincare model, but only with the Klein model, because the intersections of the lines occur outside the disk. See Figure 26.

$$A = \begin{bmatrix} 2 & -2.2 & -2.2 \\ -2.2 & 2 & -2.2 \\ -2.2 & -2.2 & 2 \end{bmatrix}$$

Note: This is not considered “hyperbolic” The definition of hyperbolic Coxeter groups requires $\mathrm{sig}(A) = (+^{k-1}0^{n-k}-)$ and $\mathrm{sig}(\hat{A}_{ii}) = (+^{j-1}0^{n-j})$

**October 20 – Assorted topics about inversions and parabolic subgroups.** Today we catch up on some scattered topics from the homework:

**Roots and the reflection sequence** Given $w \in W$, we originally defined $\mathrm{inv}(w)$ by taking $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ and setting $t_k = v_k v_{k-1} = s_{i_1} \dots s_{i_k} \dots s_{i_1}$, to get

$$\mathrm{inv}(w) := \{t \in T : t \text{ occurs an odd number of times in } t_1, \dots, t_\ell\}.$$

We then got the equivalent description,

$$\mathrm{inv}(w) = \{t \in T : \mathrm{Fix}(t) \text{ separates } D \text{ and } wD\},$$

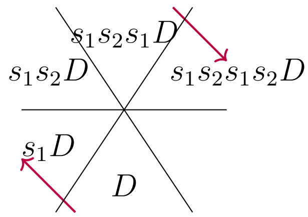
showing our original description did not depend on choice of word for $w$.

In the homework, we put $\beta_k = s_{i_1} \dots s_{i_{k-1}} \alpha_k$ so $\mathrm{Fix}(t_k) = \beta_k^\perp$. The chambers $s_{i_1} s_{i_2} \cdots s_{i_{k-1}} D$ and $s_{i_1} s_{i_2} \cdots s_{i_k} D$ are separated by $\beta_k^\perp$, with $\langle \beta_k, \ \rangle$ positive on the $s_{i_1} s_{i_2} \cdots s_{i_{k-1}} D$ side. We showed that the first time $t$ occurs as $t_{k_1}$, $\beta_{k_1}$ is a positive root, the next time $t$ occurs, $\beta_{k_1}$ is a negative root, and this continues to alternate.

**Example.** In $A_2$, looking at $s_1 s_2 s_1 s_2$, we calculate:

$$\begin{aligned} \beta_1 &= \alpha_1 \\ \beta_2 &= s_1 \alpha_2 = \alpha_1 + \alpha_2 \\ \beta_3 &= s_1 s_2 \alpha_1 = \alpha_2 \\ \beta_4 &= s_1 s_2 s_1 \alpha_2 = -\alpha_1. \end{aligned}$$

The fourth step crosses $\alpha_1$ in the opposite direction:




Using this we can see if a word is reduced. Note that the action of the $s_i$ on the $\alpha_i$ basis is easy to compute, and it is easy to see in the $\alpha_i$ basis if a root is positive or negative.

**There are many equivalent ways to describe inversions.** We have $t \in \operatorname{inv}(w)$ if and only if:

- $\beta_t$ separates $D$ and $wD$,
- $\langle \beta_t, wD \rangle \leq 0$, and
- $w^{-1}\beta_t$ is a negative root.

If $\beta$ is a postive root not a multiple of $\alpha_i$, then $s_i\beta$ is positive.

**Weak order** We define the weak order on $W$ by $u \leq v$ if $\ell(v) = \ell(u) + \ell(u^{-1}v)$. In other words, if you concatenate a reduced word for $u$ and one for $u^{-1}v$, you get a reduced word for $v$ whose prefix of length $\ell(u)$ multiplies to $u$. This is equivalent to containment of inversion sets, $u \leq v$ if and only if $\operatorname{inv}(u) \subseteq \operatorname{inv}(v)$.

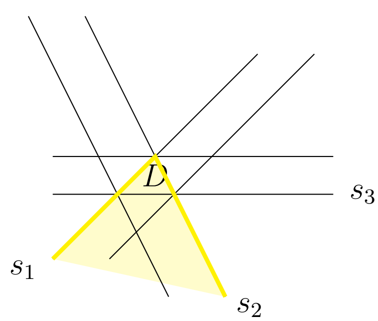
**Parabolic subgroups** Early on, we showed $wD^\circ \cap D^\circ = \emptyset$ for $w \neq 1$. We know discuss what happens for $x$ in the boundary of $D$.

Let $x \in D$. Let $I = \{i | \langle \alpha_i, x \rangle = 0\}$, and $W_I = \langle s_i \rangle_{i \in I} \subseteq W$. For $w \in W$, the following are equivalent:

- $wx \in D$,
- $wx = x$, and
- $w \in W_I$.

This is not obvious, as the conditions seem to get much stricter as we go down the list (for example, $\{w | wx \in D\}$ is not even clearly a group). Note: $W_I$ is a Coxeter group, with $\{m_{ij}\}_{i,j \in I}$.

**Example.** Letting $I = \{1, 2\}$,




the region $D$ for $W_I$ is the yellow region.

Our subgroup $W_I$ is called a *standard parabolic subgroup*. A *parabolic subgroup* is a subgroup of the form $uW_Iu^{-1}$ for some $u \in W$. We can also define these subgroups as $\{\text{Stab(x)} | \text{x} \in \text{D}\}$ and $\{\text{Stab(x)} | \text{x} \in \text{Tits(w)}\}$ respectively.

If you hate geometry, you can use this to describe triangulation of the Tits cone as an abstract simplicial complex. How? The vertices are the cosets $W/W_{[n]-\{i\}}$ with $i \in [n]$, and the faces are collections of cosets $\{u_r W_{[n]-\{i_n\}}\}$, where $\bigcap u_r W_{[n]-\{i_n\}} \neq \emptyset$.

**October 23 – Examples of Lie Groups.** A Lie group is a manifold $G$ with a group structure such that $(x, y) \mapsto xy$ is a smooth map $G \times G \to G$. By the implicit function theorem, this implies that the map $g \mapsto g^{-1}$ is also smooth.

**Examples:** We can have $G = \mathbb{R}$, with addition, or $S^1 = \mathbb{R}/\mathbb{Z}$. Passing up to higher dimensions, we can have $G = V$, a finite dimensional vector space, or $V/\Lambda$, the quotient of $V$ by a lattice. A group of the form $V/\Lambda$, with $\Lambda$ a lattice of maximal rank, is called a ***torus***. Of course, every torus is $(\mathbb{R}/\mathbb{Z})^n$, but we shouldn't think of it as coming with a specified basis.

Any group, with the discrete topology, is a 0-dimensional Lie group. For this reason, we focus on describing the connected Lie groups.

We can have $G = \mathrm{GL}_n\mathbb{R}$ or $\mathrm{GL}_n\mathbb{C}$, these are open submanifolds of $\mathbb{R}^{n^2}$ and $\mathbb{R}^{2n^2}$.

We have the orthogonal and special orthogonal groups:

$$O(n) = \{X \in \mathrm{GL}_n\mathbb{R} : XX^T = \mathrm{Id}_n\}$$
$$\mathrm{SO}(n) = \{X \in O(n) : \det X = 1\}.$$

We also have their unitary variants, which preserve the standard Hermitian form on $\mathbb{C}^n$:

$$U(n) = \{X \in \mathrm{GL}_n\mathbb{C} : X\bar{X}^T = \mathrm{Id}_n\}$$
$$\mathrm{SU}(n) = \{X \in U(n) : \det X = 1\}.$$

We'll need to compute a lot in $SU(2)$, so it is worth knowing that it has an alternate description. Set

$$I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Then

$$SU(2) = \{a + bI + cJ + dK : a^2 + b^2 + c^2 + d^2 = 1\}.$$

The main result which we will discuss (not prove!) this week is

**Theorem.** Isomorphism classes of compact connnected Lie groups are in bijection with isomorphism classes of quintuples $(W, \Lambda, \Lambda^\vee, \Phi, \Phi^\vee)$ where $\Lambda$ and $\Lambda^\vee$ are dual lattices, $\Phi \subset \Lambda$, $\Phi^\vee \subset \Lambda^\vee$ and $W$ is a finite Coxeter group acting dually on $\Lambda$ and $\Lambda^\vee$ such that $\Phi$ and $\Phi^\vee$ form roots and coroots.

The correspondence will be defined in terms of a maximal torus $T$ of $G$, meaning that $T \subset G$ is a torus, and is contained in no larger torus. There are several foundational theorems on maximal tori, such as:

**Theorem.** Any compact connected Lie group has a maximal torus $T$, and any other maximal torus is conjugate to $T$. We have $G = \bigcup_{g \in G} gTg^{-1}$, so every conjugacy class of $G$ meets $T$.

We will have $\Lambda = \mathrm{Hom}(T, S^1)$ and $\Lambda^\vee = \mathrm{Hom}(S^1, T)$. Or, for those who like algebraic topology, $\Lambda = H^1(T, \mathbb{Z})$ and $\Lambda^\vee = H_1(T, \mathbb{Z})$. The dual pairing between $\mathrm{Hom}(T, S^1)$ and $\mathrm{Hom}(S^1, T)$ occurs as follows: Let $\gamma : S^1 \to T$ and $\chi : T \to S^1$. Then $\chi \circ \gamma : S^1 \to S^1$ must be of the form $\theta \mapsto m\theta$ for some $m$; we put $\langle \chi, \gamma \rangle = m$.

The normalizer of $T$, $N(T)$, is defined to be $\{g \in G : gTg^{-1} = T\}$. Set $W = N(T)/T$. So $W$ acts on $T$ by conjugation. Clearly, for $g \in N(T)$ and $t \in T$, the elements $t$ and $gtg^{-1}$ are conjugate. In fact, the reverse is true as well

**Theorem.** The set of conjugacy classes of $G$ is $W/T$.

The rest of today is spent on examples.

**Example.** Let $G = T$. Then $N(T) = T$ and $W$ is trivial.

**Example.** Let $G = U(n)$. A maximal torus is the diagonal matrices $\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$. We have $\Lambda$ and $\Lambda^\vee$ naturally identified with $\mathbb{Z}^n$. So $N(T)$ is the matrices which conjugate diagonal matrices to diagonal matrices; these can be seen to be matrices of the form

$$\begin{bmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ * & 0 & 0 & 0 \end{bmatrix}$$

where the $*$ are roots of unity and we may take any permutation. In other words, $N(T) = S_n \ltimes (S^1)^n$ and $W = S_n$.

**Example.** Let $G = SU(n)$. Then $T = \{\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) : \sum \theta_i = 0\}$. The lattice $\Lambda^\vee = \mathrm{Hom}(S^1, T)$ is $\{(k_1, \dots, k_n) : \sum k_i = 0\}$, specifically, the maps are $\theta \mapsto \mathrm{diag}(e^{ik_1\theta}, e^{ik_2\theta}, \dots, e^{ik_n\theta})$. The dual lattice $\Lambda$ is $\mathbb{Z}^n/\mathbb{Z}(1, 1, \dots, 1)$, since the character $\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mapsto \sum \ell_i\theta_i$ is unaltered by adding the same integer to all the $\ell_i$. We still have $W = S_n$, acting on $\Lambda$ and $\Lambda^\vee$ in the standard way.

A word of warning: The sequence

$$1 \to T \to N(T) \to S_n \to 1$$

is no longer semidirect. Taking $n = 2$, the non-identity element of $S_2$ would have to lift to a matrix of the form $\left[\begin{smallmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{smallmatrix}\right]$, but such a matrix squares to $-\mathrm{Id}_2$, not $\mathrm{Id}_2$.

**Example.** Let $\mathbb{P}U(n) = U(n)/\mathbb{C}^*$, with $\mathbb{C}^*$ embedded as the center $\mathrm{diag}(e^{i\phi}, e^{i\phi}, \dots, e^{i\phi})$. This time we have $\Lambda^\vee = \mathbb{Z}^n/\mathbb{Z}(1, 1, \dots, 1)$ and $\Lambda = \mathbb{Z}^n \cap (1, 1, \dots, 1)^\perp$. Note that we can write $\mathbb{P}U(n)$ as $\mathrm{SU}(n)/\langle \mathrm{diag}(\zeta, \zeta, \dots, \zeta)\rangle$ where $\zeta$ is a primitive $n$-th root of unity. We can interpolate between $\mathrm{SU}(n)$ and $\mathbb{P}U(n)$ by quotienting by various subgroups of the cyclic group of order $n$, and get other lattices which are preserved by $S_n$.

**Example.** Now let's look at $\mathrm{SO}(m)$. A maximal torus looks like

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 & \cdots & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 & \cdots & 0 & 0 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cos\theta_n & -\sin\theta_n \\ 0 & 0 & 0 & 0 & \cdots & \sin\theta_n & \cos\theta_n \end{bmatrix} \text{ in } \mathrm{SO}(2n)$$

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cos\theta_n & -\sin\theta_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & \sin\theta_n & \cos\theta_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \text{ in } \mathrm{SO}(2n+1).$$

The normalizer $N(T)$ in

$$\begin{bmatrix} 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{bmatrix} \text{ in } \mathrm{SO}(2n)$$

$$\begin{bmatrix} 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix} \text{ in } \mathrm{SO}(2n+1)$$

where each $2 \times 2$ block is in $O(2)$.

We must make sure that the determinant is 1. In $\mathrm{SO}(2n+1)$, this just amounts to choosing the sign in the lower right accordingly. We get $N(T) = S_n \ltimes O(2)^n$ so $N(T)/T = S_n \ltimes (\mathbb{Z}/2)^n$. This is the Coxeter group of type $B_n$. In $\mathrm{SO}(2n)$ we have to make sure that an even number of the boxed entries are in the non-identity component of $O(2)$, so $N(T)/T = S_n \ltimes (\mathbb{Z}/2)^{n-1}$. This is the Coxeter group of type $D_n$.

It isn't clear yet why we are calling $\mathrm{SO}(2n+1)$ type $B$ and not type $C$; we'll discuss this later. There is also a family of Lie groups realizing type $C$. We didn't get to this in class, but I'll put it in the notes here:

**Example.** Let $\mathbb{H}$ be the ring of quaternions and let $\overline{a + bI + cJ + dK} = a - bI - cJ - dK$. Put

$$U(n, \mathbb{H}) = \{X \in \mathrm{Mat}_{n\times n}(\mathbb{H}) : X\overline{X}^T = \mathrm{Id}_n\}.$$

A maximal torus is matrices of the form $\mathrm{diag}(\cos\theta_1 + I\sin\theta_1, \cos\theta_2 + I\sin\theta_2, \cdots, \cos\theta_n + I\sin\theta_n)$. The normalizer is permutation-like matrices where the nonzero entries are in $\{\cos\theta + I\sin\theta\} \cup \{J\cos\theta + K\sin\theta\}$. We have $W = S_n \ltimes (\mathbb{Z}/2)^n$. We'll see later why this should be called $C_n$.

We close with an example which shows that we really need the extra data of $\Phi$ and $\Phi^\vee$, which we have not discussed yet:

**Example.** Consider the groups SU(2) and SO(3). They are not isomorphic; for example, the center of SU(2) is $\pm \mathrm{Id}_2$, while the center of SO(3) is trivial. In fact, $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\pm \mathrm{Id}_2$.

In both these groups, a maximal torus is 1-dimensional, namely

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In both cases, $N(T)/T$ is the cyclic group of order 2, generated by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

respectively. So in both cases, $W \cong \mathbb{Z}/2$ and $\Lambda \cong \Lambda^\vee \cong \mathbb{Z}$ with $W$ acting by negation. We need to add extra data to distinguish them!

**October 25 – Lie algebra, exponential, and Lie bracket.** Let $G$ be a Lie group, $\mathfrak{g} = T_e G$.

**Example.**
- When $G = \mathbb{R}$, $\mathfrak{g} = \mathbb{R}$.
- When $G = \mathbb{R}/\mathbb{Z}$, $\mathfrak{g} = \mathbb{R}$.
- When $G = V/\Lambda$, $\mathfrak{g} = V$.
- When $G = O(n)$ (or $SO(n)$), which is $\{X \in \mathrm{GL}(n), XX^T = Id\}$, we have
$$(1+\epsilon J)(1+\epsilon J)^T = 1 + \epsilon(J + J^T) + O(\epsilon^2).$$
Then the Lie algebra is $\mathfrak{o}(n) = \mathfrak{so}(n) = \{J \in Mat_{n\times n}(\mathbb{R}), J + J^T = 0\}$, the set of skew-symmetric matrices.
- When $G = \mathrm{GL}_n(\mathbb{R})$, $\mathfrak{gl}_n\mathbb{R} = Mat_{n\times n}(\mathbb{R})$. And since
$$\det(1+\epsilon X) = 1 + \epsilon \sum_{i=1}^n X_{ii} + O(\epsilon^2),$$
we have
$$\mathfrak{sl}_n = \{X \in Mat_{n\times n}(\mathbb{R}) : \sum_{i=1}^n X_{ii} = 0\}.$$
- When $G = U(n) = \{X \in Mat_{n\times n}(\mathbb{C}), X\overline{X}^T = Id\}$, we have
$$\mathfrak{u}(n) = \{J \in Mat_{n\times n}(\mathbb{C}) | J + \overline{J}^T = 0\}$$
$$\mathfrak{su}(n) = \{J | J + \overline{J}^T = 0,\ \mathrm{Tr}(J) = 0\}.$$
In particular, for $SU(2) = \{a + bI + cJ + dK | a^2+b^2+c^2+d^2 = 1\}$,, we have
$$\mathfrak{su}(2) = \{pI + qJ + rK\}.$$

**Lie Bracket** The multiplication $\mu : T_{e,e}(G\times G) \to T_e G$ induces the additive structure $\mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$. So we could define the Lie bracket to be a bilinear skew symmetric map

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

Defined by the following property: If $\vec{u}$, $\vec{v} \in \mathfrak{g}$, and $\gamma$ and $\delta : (-1, 1) \to G$ are smooth paths with $\gamma(0) = \delta(0) = e$, $\gamma'(0) = \vec{u}$ and $\delta'(0) = \vec{v}$ then we have

$$\gamma(s)\delta(t)\gamma(s)^{-1}\delta(t)^{-1}st = [\vec{v}_1, \vec{v}_2] + O(|s^2||t| + |s||t^2|).$$

For $G \subseteq \mathrm{GL}_n$, $\mathfrak{g} \subseteq \mathfrak{gl}_n = Mat_{n\times n}$, then the Lie bracket on $\mathfrak{g}$ is given by

$$[X, Y] = XY - YX.$$

Let's see this with a computation in coordinates:

$$n(1+sX)(1+tY)(1+sX)^{-1}(1+tY)^{-1} = (1+sX)(1+tY)(1-sX+s^2X^2-\cdots)(1-tY+t^2Y^2-\cdots)$$
$$= 1 + (XY - YX)st + \cdots .$$

**Exponential** We define $\exp : \mathfrak{g} \to G$, such that for a smooth path$\gamma : (-1, 1) \to G$ with $\gamma(0) = e$, $\gamma'(0) = \vec{v}$, the map is given by

$$\exp \vec{v} = \lim_{n\to\infty} \gamma(\frac{1}{n})^n.$$

If $G = \mathbb{R}$, $a \in \mathfrak{g} = \mathbb{R}$, and $\gamma(t) = at$, then we have

$$\lim_{n\to\infty} \gamma(\frac{1}{n})^n = a,$$

so $\exp : \mathbb{R} \to \mathbb{R}$ is the identity. And exp gives an obvious quotient $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

When $\mathbb{G} = \mathbb{R}^{\times}_{>0}$, $\mathfrak{g} = \mathbb{R}$, $\gamma(t) = 1 + at$, we have

$$\lim_{n\to\infty} (1 + \frac{a}{n})^n = e^a.$$

More generally, if $G \subseteq \mathrm{GL}_n$, and $X \in \mathfrak{g} \subseteq Mat_{n\times n}$, we have

$$\lim_{n\to\infty} (1 + tX)^n = \sum_{k=0}^{\infty} \frac{X^k}{k!} t^k.$$

We note that for a given $\Lambda^\vee \subset V^\vee$, $T = V^\vee/\Lambda^\vee$, and $\mathfrak{t} = V^\vee$. Then $\Lambda^\vee$ is the kernel of $\exp : \mathfrak{t} \to T$, so $\Lambda$ naturally sits in $\mathfrak{t}^\vee$.

**Adjoint action** For given Lie group $G$, it can act on itself by conjugation as

$$g : h \mapsto hgh^{-1},$$

which fixes $e$. So by taking the differential, we get the action of $G$ on $T_eG = \mathfrak{g}$. This is called the adjoint action

$$ad_g : \mathfrak{g} \to \mathfrak{g}.$$

When $G = \mathrm{GL}_n$, $\mathfrak{g} = Mat_{n\times n}$, we have

$$ad_g(X) = gXg^{-1}.$$

If $G$ is compact, the action will preserve a positive definite symmetric bilinear form on $\mathfrak{g}$. If $T$ is a torus of $G$, then $\mathfrak{t} \subset \mathfrak{g}$ inherits a dot product $\cdot$. And Induced from the conjugate action, we get the action of $W = N(T)/T$ on $T$, and on $\mathfrak{t}$ correspondingly, which preserves the $\cdot$ and $\ker(\exp : \mathfrak{t} \to T)$.

**The action of $T$ on $\mathfrak{g}$** Since $T$ is maximal, the fixed subspace of $\mathfrak{g}$ is $\mathfrak{t}$. We then have the decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\chi\in(\Lambda-\{0\})/\pm 1} \mathfrak{g}_\chi,$$

where $\Lambda = \mathrm{Hom}(T, S^1)$, $\theta \in \mathfrak{t}$ acts on $\mathfrak{g}_\chi$ by

$$\begin{pmatrix} \cos\chi(\theta) & -\sin\chi(\theta) \\ \sin\chi(\theta) & \cos\chi(\theta)) \end{pmatrix}.$$

The character $\chi$ is only defined modulo $\pm 1$ because $\mathfrak{g}_\chi$ doesn't come with a natural orientation, so we can't distinguish between a clockwise rotation by $\theta$ and a counterclockwise one.

We let $\Phi \subset \Lambda \backslash \{0\}$ to be subset consisting of those $\chi$ such that $\mathfrak{g}_\chi \neq 0$. So $\Phi$ consists of pairs $\pm\chi$.

**Example.**

- When $G = T$, $\mathfrak{g} = \mathfrak{t}$, $\Phi = \varnothing$.
- When $G = SU(2)$, $T = \{\cos\theta + I\sin\theta\} = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\}$, where elements in $T$ can also be written as
$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$$
Then we have
$$\mathfrak{g} = \{pI + qJ + rK\}, \mathfrak{t} = \{pI\}.$$
And by computing the conjugation explicitly, we have
$$\begin{aligned}(\cos\theta + I\sin\theta)J(\cos\theta - I\sin\theta) &= (\cos^2\theta - \sin^2\theta)J + (2\sin\theta\cos\theta)K \\ &= \cos 2\theta J + \sin 2\theta K. \\ (\cos\theta + I\sin\theta)K(\cos\theta - I\sin\theta) &= -\sin 2\theta J + \cos 2\theta K.\end{aligned}$$
So we have
$$\Phi = \{\pm 2\} \subset \Lambda = \mathbb{Z}.$$
- When $G = SO(3)$, $\mathfrak{g} = \{\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ y & -z & 0 \end{pmatrix}\}$, we let $T = \{\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Then by computing the conjugation, we get
$$\begin{aligned}&\begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ y & -z & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x & y\cos\theta + z\sin\theta \\ -x & 0 & y\sin\theta + z\cos\theta \\ * & * & 0 \end{pmatrix}\end{aligned}$$
So from this, we see $\Phi = \{\pm 1\} \subset \Lambda = \mathrm{Hom}(T, S^1)$.
- When $G = U(n)$, $T$ is given by the set of $diag(e^{i\theta_1}, \ldots, e^{i\theta_n})$, and Lie algebra is
$$\mathfrak{g} = \{\begin{pmatrix} iz_1 & & & x_{pq} + iy_{pq} \\ & iz_2 & & \\ & & \ddots & \\ -x_{pq} + iy_{pq} & & & iz_n \end{pmatrix}.$$

And like before we compute the conjugation:

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \begin{pmatrix} iz_1 & & & x_{pq}+iy_{pq} \\ & iz_2 & & \\ & & \ddots & \\ -x_{pq}+iy_{pq} & & & iz_n \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n} \end{pmatrix},$$

which gives $e^{i\theta_q}(x_{pq}+iy_{pq})e^{-\theta_1}$. So we get

$$\Phi = \{e_p - e_q\} \subset \mathbb{Z}^n.$$

**October 29 - Co-roots in Lie groups.**

**Setup.** Let $G$ be a compact Lie Group with Lie Algebra $\mathfrak{g}$. Let $T$ be it's maximal torus, with Lie algebra $\mathfrak{t}$. We have a $G$-action on $\mathfrak{g}$ and thus a $T-$action on $\mathfrak{g}$.

Recall from last time that $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\chi\in\Phi/\pm 1} \mathfrak{g}_\chi$ and that we defined $\Phi \subseteq \Lambda = \mathrm{Hom}(T, S^1)$.

Then, we have $W = N(T)/T$ acting on $\mathfrak{t}, \Lambda, \Phi$

However, this picture doesn't include coroots. Today we will see how to incorporate them.

To begin with, fix some $\pm\chi \in \Phi$. We know $\mathfrak{g}_\chi \simeq \mathbb{R}^2$ with $T$ acting by rotation by some $\theta$.

Let $J_0$, $K_0$ be an orthogonal basis for $\mathfrak{g}_\chi$. Then, let $I_0 = [J_0, K_0]$. For any $t \in T$, $I_0 = [t\cdot J_0, t\cdot K_0]$ (since $t$ acts by rotation on the 2D space).

Thus, $t \cdot I_0$, so $I_0 \in \mathfrak{t}$. Then, $[I_0, J_0] = \chi(I_0)K_0$ and $[I_0, K_0] = -\chi(I_0)J_0$.

Now recall that $\mathrm{Hom}(T, S^1) \subseteq \mathfrak{t}^\vee$ because for $\theta \in \mathfrak{t}$ and $\chi \in \mathfrak{t}^\vee$, we have that $\chi(\theta \mod \Lambda^\vee) = \langle \chi, \theta\rangle \mod \mathbb{Z}$.

**Example.** For $T = \mathbb{R}^n/\mathbb{Z}^n$, $\mathbb{Z}^n = \Lambda^\vee \subseteq \mathfrak{t} = \mathbb{R}^n$ and $\mathbb{Z}^n = \Lambda \subseteq \mathfrak{t}^\vee = \mathbb{R}^n$

This is because we have compatibility $\mathrm{ad}_{e^x} = \exp[x, -]$ since

$$e^x y e^{-x} = \sum_{n=0}^{\infty} \frac{\overbrace{[X, \ldots, [X, [X}^{n}, Y]] \ldots]}{n!}$$

Thus, $\mathrm{Span}_{\mathbb{R}}(I_0, J_0, K_0)$ is a sub Lie-algebra of $\mathfrak{g}$.

We claim without proof that $I_0 \neq 0$ and $\chi(I_0) > 0$, so this is a basis. (See below for comments on this.)

We put $I = \frac{2I_0}{\chi(I_0)}$, $J = \sqrt{\frac{2}{\chi(I_0)}}J_0$ and $K = \sqrt{\frac{2}{\chi(I_0)}}K_0$. Then, we get

$$\begin{array}{ccc} \mathfrak{su}(2) & \hookrightarrow & \mathfrak{g} \\ \uparrow & & \uparrow \\ \mathbb{R}I & \hookrightarrow & \mathfrak{t} \end{array}$$

Since $\mathfrak{su}(2)$ is simply connected, we get

$$\begin{array}{ccc} SU(2) & \hookrightarrow & G \\ \uparrow & & \uparrow \\ \exp(\mathbb{R}I) \simeq S^1 & \hookrightarrow & T \end{array}$$

So we get a pair of maps $\pm\beta^\vee \in \Lambda^\vee = \mathrm{Hom}(S^1, T)$. These are our **coroots**.

Thus, we have constructed $(W, \Lambda, \Lambda^\vee, \Phi, \Phi^\vee)$.

**Question.** Where do the reflections come from that make this a reflection group?

Think of $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Then, we note that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} I \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -I$.

**Example.** $SO(2n) \rightsquigarrow (D_n, \Lambda, \Lambda^\vee, \Phi, \Phi^\vee)$, where $(\pm e_i \pm e_j) = \Phi \subseteq \Lambda = \mathbb{Z}^n$ and $(\pm e_i \pm e_j) = \Phi^\vee \subseteq \Lambda^\vee = \mathbb{Z}^n$.

Then, $\Phi \subseteq \mathbb{Z}^n \subseteq (\mathbb{Z}\Phi^\vee)^\vee$, with both $\subseteq$ of index 2. So this means that there are two other options for the group $D_n$.

$\frac{SO(2n)}{\pm \mathrm{Id}_{2n}} \longleftarrow SO(2n) \longleftarrow \mathrm{Spin}(2n)$

(Spin$(2n)$ is a $2:1$ cover of $SO(2n)$).

Without proof we claim that $G/_{\mathrm{conj}}G \simeq T/W$ (where ${}_{\mathrm{conj}}G$ is $G$ acting by conjugation) and that $\mathfrak{g}/_{\mathrm{ad}}G \simeq \mathfrak{t}/W$.

**Example.** We can apply these two to various Lie Groups to get various results about matrixes. For example, the first equation applied to unitary group: Every unitary matrix $g$ is $U \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} U^{-1}$, where $(\theta_1, \ldots, \theta_n)$ unique up to permutation.

Applying the second to the corresponding Lie algebra gives that every skew Hermetian matrix $X$ is $U \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_n \end{pmatrix} U^{-1}$, where $(\lambda_1, \ldots, \lambda_n)$ unique up to permutation.

**Example.** Now, we do the same with groups of type $D$:

In $SO(2n)$, every $g = U \begin{pmatrix} \square & & \\ & \ddots & \\ & & \square \end{pmatrix} U^{-1}$, where each $\square$ is $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ where the $(\theta_1, \ldots, \theta_n)$ unique up to permutation and reversing sign of an even number of them.

The Lie Algebra version is that every skew-symmetric $2n \times 2n$ matrix $X$ is of the form $g = U \begin{pmatrix} \square & & \\ & \ddots & \\ & & \square \end{pmatrix} U^{-1}$, where each $\square$ is $\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$ where the $(\lambda_1, \ldots, \lambda_n)$ unique up to permutation and reversing sign of an even number of them. In particular, $\lambda_1 \cdots \lambda_n$ is well-defined under SO$(2n)$ conjugacy – it is called the Pfaffian of $X$.

We remark on the claim that $I_0 \neq 0$ and $\chi(I_0) > 0$. If $I_0 = 0$, so $[J_0, K_0]$ commute, then the simply connected group with Lie algebra $\mathrm{Span}(I_0, J_0, K_0)$ is $\mathrm{Span}(I_0) \ltimes \mathrm{Span}(J_0, K_0)$ where $\theta I_0$ acts by rotation by $\theta$. This has no compact quotients. If $I_0 \neq 0$ and $\chi(I_0) \leq 0$, then the adjoint action of $\exp(tK_0)$ on $\mathrm{Span}(I_0, J_0)$ is $\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right]$. These actions do not preserve a positive definite inner product.

**October 30 – Examples and notation for invariants.** We fix a ring $R$, and a finite Coxeter group $W$. Let $S = R^W$ be the subring of $W$ invariants. We consider two settings.

First, let $V$ be the reflection group of $W$, while

$$R = Sym^\bullet(V) = \mathbb{R} \oplus V \oplus Sym^2 V \oplus \ldots$$

consists of the polynomial functions on $V^\vee$. Again, $S = R^W$. Note that

$$\mathfrak{g}/\operatorname{Ad}\mathfrak{g} \cong t/W \cong V^\vee/W.$$

**Example.** Working with the $A_{n-1} \cong S_n$-action on $\mathbb{R}^n$, $\mathbb{R}[x_1,\ldots,x_n]^{S_n} = \mathbb{R}[e_1,\ldots,e_n]$ by the Fundamental Theorem of Symmetric Functions, where

$$e_k \sum_{i_1<i_2<\ldots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

**Example.** Working with $A_{n-1}$, set

$$V = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}^\perp, \quad V^\vee = \mathbb{R}^n/\mathbb{R}\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}.$$

We then have

$$R = \mathbb{R}[x_1,\ldots,x_n]/(x_1+x_2+\cdots+x_n), \quad S = \mathbb{R}[e_1,\ldots,e_n]/(e_1) \cong \mathbb{R}[e_2,\ldots,e_n].$$

**Example.** Working with $B_n = C_n$, $R = \mathbb{R}[x_1,\ldots,x_n]$. Note that $W$ acts by permuting and negating the $x_i$'s, so that $f_k = e_k(x_1^2,\ldots,x_n^2) \in S$. Indeed, $S = \mathbb{R}[f_1,\ldots,f_n]$.

**Example.** Working with $D_n$, which is an index two subgroup of $B_n$ $(C_n)$ preserving $x_1x_2\cdots x_n$, we have $S = \mathbb{R}[f_1,\ldots,f_{n-1},e_n]$.

**Example.** Working with $I_2(m)$, $S = \mathbb{R}[x^2+y^2, F]$ where $F$ is a degree $m$ polynomial.

These examples illustrate a general result that we will eventually prove:

**Theorem.** (Chevalley-Shephard-Todd) $S$ is a polynomial ring on $\dim V$ generators.

Now we move to a second setting/question: given a cystallographic Coxeter group $W$, and an invariant lattice $\Lambda$, we set

$$\widehat{R} = \mathbb{R}[\Lambda], \quad \widehat{S} = \widehat{R}^W.$$

These are the functions on $G/\operatorname{Ad}G \cong T/W$. Note that $A_{n-1}$ acts on $\mathbb{Z}^n$, and

$$\mathbb{R}[x_1^\pm,\ldots,x_n^\pm]^{S_n} = \mathbb{R}[e_1,e_2,\ldots,e_{n-1},e_n^\pm].$$

Note that $A_{n-1}$ acts on $\Lambda = \mathbb{Z}^n/\mathbb{Z}\cdot\begin{bmatrix}1\\\vdots\\1\end{bmatrix}$. In this case, we have

$$\widehat{R} = \mathbb{R}[x_1^\pm,\ldots,x_n^\pm]/(x_1x_2\cdots x_n - 1).$$

$$\widehat{S} = \mathbb{R}[e_1,e_2,\ldots,e_{n-1},e_n^\pm]/(e_n-1) \cong \mathbb{R}[e_1,\ldots,e_{n-1}].$$

By way of contrast, we note that $A_{n-1}$ acts on $\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}^\perp \cap \mathbb{Z}^n$, in which case

$$\widehat{R} = \mathbb{R}[x_i/x_j \colon 1 \le i \ne j \le n] \subseteq \mathbb{R}[x_1^\pm,\ldots,x_n^\pm],$$

while $\widehat{S}$ consists of the degree zero elements in $\mathbb{R}[e_1,\ldots,e_{n-1},e_n^\pm]$.

The answer is nicest for the weight lattice $\Lambda = (\mathbb{Z}\Phi^\vee)^\vee$. In this case, we can consider the vector
$$\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta = \sum w_i,$$
which satisfies $\langle \rho, \alpha_i\rangle = 1$ for $i = 1, \ldots, n$.

Working with $C_n$ and the lattice $\Lambda = \mathbb{Z}^n$, $\widehat{R} = \mathbb{R}[x_1^\pm, \ldots, x_n^\pm]$. The group $W$ acts by permutations and reciprocals. This really is not the same as the $W$-action on $R$ by permutations and negation. Note that in working with $\Phi = \{\pm 2e_k, \pm e_i \pm e_j\}$,
$$\Phi^\vee = \{\pm e_k, \pm e_i \pm e_j\}, \text{ and } (\mathbb{Z}\Phi^\vee)^\vee = \mathbb{Z}^n.$$

**The element** $\rho$. This is defined as $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$. For all $i$, we have $\langle \rho, \alpha_i^\vee\rangle = 1$. Proof: We know that $s_i$ permutes $\Phi^+ \setminus \{\alpha_i\}$, so $\langle \frac{1}{2}\sum_{\beta\in\Phi^+\setminus\{\alpha_i\}}\beta,\ \alpha_i^\vee\rangle = 0$ and $\langle \rho, \alpha_i^\vee\rangle = (1/2)\langle \alpha_i, \alpha_i^\vee\rangle = (1/2)\cdot 2 = 1$.

We can use the second formula $\langle \rho, \alpha_i^\vee\rangle = 1$ to define $\rho$ for infinite groups.

We have
$$\begin{array}{rcl} s_i\rho & = & \rho - \alpha_i \\ s_{i_1}s_{i_2}\rho & = & \rho - \alpha_{i_1} - s_{i_1}\alpha_{i_2} \\ \vdots & & \vdots \\ s_{i_1}s_{i_2}\cdots s_{i_\ell}\rho & = & \rho - \sum_{k=1}^{\ell} s_{i_1}\cdots s_{i_k}\alpha_k \\ \text{so we deduce } w\rho & = & \rho - \sum_{t\in\mathrm{inv}(w)}\beta_t \end{array}$$

**November 1 – Commutative algebra background.** We assume all rings are commutative with 1. A ring $R$ is called Noetherian if for any finitely generated $R-$module $M$, and $M' \subset M$ any submodule, $M'$ is finitely generated.

**Remark.** The following wasn't said until November 3 but should have been said here: If $M$ is a finitely generated $R$-module, and $g_\alpha$ is a possibly infinite list of generators for $M$ as an $R$-module, then there is a finite subset of the $g_\alpha$ which generate $M$ as an $R$-module. **Proof:** Let $M = \langle h_1, \ldots, h_k\rangle$ and write $h_i = \sum_{\alpha\in A_i} f_{i\alpha}g_\alpha$ where $A_i$ is some finite set. Then $M$ is generated by $\bigcup_i \{g_\alpha : \alpha \in A_i\}$.

**Theorem. (Hilbert Basis Theorem)** Let $k$ be a field, then the polynomial ring $k[x_1, x_2, \ldots, x_n]$ is Noetheran.

**Corollary.** Any finitely generated $k$ algebra is Noetherian.

Indeed, if $A$ is a finitely generated by elements $\theta_1$, $\theta_2$, $\ldots$, $\theta_n$ in $A$, then we get a homomorphism $k[x_1, x_2, \ldots, x_n] \to A$ by sending $x_i$ to $\theta_i$. this map makes $A-$modules into $k[x_1, x_2, \ldots, x_n]-$modules.

$R$ is called ***graded*** if it is equipped with a decomposition $R = \oplus_{i=0}^\infty R_i$ as an additive group and $R_iR_j \subset R_{i+j}$. A $R-$module $M$ is called ***graded*** if $M = \oplus_{i=0}^\infty M_i$ and $R_iM_j \subset M_{i+j}$.

An ideal $\mathfrak{m}$ is maximal if $R/\mathfrak{m}$ is a field. A ring $R$ is called ***local*** if it has precisely one maximal ideal. A graded ring $R$ is called ***graded local*** if it has precisely one graded maximal ideal.

There are two main ways to get graded local rings:

- $R_0 = k$ Then $\mathfrak{m} = \oplus_{i=1}^\infty R_i$, such as $k[x_1, x_2, \ldots, x_n]$ with degree $x_i = 1$.
- $R$ local and $R = R_0$.

Let $R$ be a graded ring, and $m$ a homogenous maximal ideal, then $R$ is graded local with maximal ideal $m$ iff for any $x \in R_0$, exactly one of the following holds: $x \in m$ or $x$ is a unit.

**Theorem. (Nakayama Lemma)** Let $R$ be a graded local ring with the graded maximal ideal $\mathfrak{m}$, $M$ a finitely generated graded $R$-module. Let $k = R/m$ be the residue field. Put $V = M \otimes_R k = M/\mathfrak{m}M$; this is a $k$ vector space. Let $e_1, \ldots, e_n$ be homogenous basis of $V$, and lift $e_i$ to a homogenous element $\tilde{e_i} \in M$. Then $\tilde{e_1}, \ldots, \tilde{e_n}$ generate $M$ as an $R-$module.

*Proof.* Let $M$ be generated by $\tilde{f}_1, \ldots, \tilde{f}_q$ as an $R-$module, without loss of generality, we can take $\tilde{f}_i$ to be homogenous. Let $f_j$ be the image of $\tilde{f}_j$ in $V$, so we can write $f_j = \sum_{i=1}^n c_{ij} e_i$. So in $M$, we have $\tilde{f}_j = \sum \tilde{c_{ij}} \tilde{e}_i + \sum \tilde{d_{kj}} \tilde{f}_k$, where $\tilde{c_{ij}} \in R$ are homogenous and $\tilde{d_{kj}} \in m$. We rewrite this in the matrix form:

$$[\tilde{f}] = [\tilde{c}][\tilde{e}] + [\tilde{d}][\tilde{f}] \text{ so } ([Id] - [\tilde{d}])[\tilde{f}] = [\tilde{c}][\tilde{e}].$$

Note that $\det([Id] - [\tilde{d}])$ is homogenous and, is 1 modulo $\mathfrak{m}$. So $det([Id] - [\tilde{d}])$ is a unit, and $([Id] - [\tilde{d}])$ is invertible, so we have $[\tilde{f}] = (Id - [\tilde{d}])^{-1}[\tilde{c}][\tilde{e}]$. This shows that the $\tilde{f}_j$ are in the $R$-module generated by the $\tilde{e}_i$, so the $\tilde{e}_i$ generate $M$. □

Our goal for next time:

**Theorem.** If $G$ is a finite group acting on a finitely dimensional vector space $V$ over a field $k$, and let $R = Sym^\bullet V$ be the symmetric algebra, then the ring of invariants $R^G$ is finitely generated.

**November 3 – Commutative algebra of rings of invariants.** Let $R$ be a $k$-algebra, $G$ a finite group acting on $R$. Let $S = R^G$.

**Theorem** (Noether)**.** If $R$ is finitely generated as a $k$-algebra, then so is $S$.

*Proof.* Let $\theta_1, \cdots, \theta_n$ generate $R$ as a $k$-algebra. Let $e_k^i$ be the $k^{th}$ elementary symmetric function of $\{g \cdot \theta_i\}$. We have $e_1^i = \sum_g g\theta_i$, $e_2^i = \sum_{g,h}(g\theta_i)(h\theta_i)$, and so on. So all $e_k^i \in S$. Let $T$ be the sub-$k$-algebra of $S$ generated by the $e_k^i$. Observe that $\prod_{g \in G}(\theta_i - g\theta_i) = 0$. So $\theta_i^{|G|} - e_1^i \theta_i^{|G|-1} + \cdots \pm e_{|G|}^i = 0$ and we see that $\theta_i^{|G|}$ is in $\text{Span}_T(\theta_i^{|G|-1}, \cdots, \theta_i, 1)$. So any $\theta_i^N$ is in $\text{Span}_T(\theta_i^{|G|-1}, \cdots, \theta_i, 1)$. So any monomial $\theta_1^{N_1}\theta_2^{N_2} \cdots \theta_n^{N_n}$ is in $\text{Span}_T(\theta_1^{a_1} \cdots \theta_n^{a_n} \ : \ a_i \leq |G| - 1)$. So $\{\theta_1^{a_1} \cdots \theta_n^{a_n} | a_i \leq |G| - 1\}$ span $R$ as a $T$-Module.

The $k$-algebra $T$ is finite generated by its definition, so it's Noetherian by the Hilbert Basis Theorem. We have just showed $R$ is a finitely generated $T$-module. Now, $S$ is a sub-$T$-module of $R$, so, by the Noetherianity property, $S$ is generated by some $\beta_1, \ldots, \beta_k$ as a $T$-module. So $S$ is generated by the $e_k^i$ and the $\beta_j$. □

In the course of this proof, we showed that $R$ is a finitely generated $S$-module, which will be important in its own right.

We now give a second proof, due to Hilbert. This time, assume $R$ is a finitely generated graded ring with $R_0 = k$, and assume $|G|$ is not divisible by char$(k)$. Assume that the action of $G$ fixes $k$ and preserves the grading.

Then $S = R^G$ has a grading $S = \oplus_{i \geq 0} S_i$, where $S_i = R_i^G$ and $S_0 = k$. Write $R_+ = \oplus_{i>0} R_i$ and $S_+ = \oplus_{i>0} S_i$. Write $J = RS_+$, the ideal generated by $S_+$ in $R$.

**Example.** If $R = k[x_1, x_2]$ and $G = S_2$ with char $k \neq 2$, then $S = k[x_1 + x_2, x_1x_2]$ and $J = \langle x_1 + x_2, x_1x_2 \rangle$ in $k[x_1, x_2]$.

By Hilbert's Basis theorem, there exist $g_1, \ldots, g_n$ generating $J$ as an ideal. We may take the $g_i$ homogeneous and in $S_+$. Let $P = k[g_1, \cdots, g_n] \subseteq S$. We claim that $P = S$.

*Proof.* We show $P_i = S_i$ by induction on $i$.

**Base case:** For $i = 0$, we have $P_0 = s_0 = k$.

Let $f \in S_i$ for $i > 0$. Since $f$ is in $S_+$ it is in $J$. So $f = \sum g_i h_i$ for some $h_i \in R$. We can assume $g_j$ and $h_j$ are homogeneous with deg $f$ = deg $g_j$ + deg $h_j$. So $f = \frac{1}{|G|}\sum_{\sigma\in G}\sum_j \sigma(g_j)\sigma(h_j) = \frac{1}{|G|}\sum_{\sigma\in G}\sum_j g_j\sigma(h_j) = \sum_j g_j \frac{\sum_{\sigma\in G}\sigma(h_j)}{|G|}$. We have $g_j \in P$ and $\frac{\sum_{\sigma\in G}\sigma(h_j)}{|G|} \in S$. By the induction hypothesis, this is in $P$. So $f \in P_i$. ☐

In the course of the proof, we have shown that any list of homogenous elements of $S_+$ which generate $RS_+$ as an $R$-module also generate $S$ as a $k$-algebra. By Nakayama's lemma, generators of $RS_+$ as an $R$-module correspond to generators of $(RS_+)\otimes_R R/R_+ = (RS_+)/(R_+S_+)$ as a $k$-vector space.

Let's think about $R$ as an $S$-module. By Noether's proof, we know it's a finitely generated $S$-module. If $R$ is a domain, then so is $S$. It is easy to check that $\operatorname{Frac} S = (\operatorname{Frac} R)^G$. If $G$ is a finite group acting on a field $L$ and every $g \neq e$ in $G$ acts non-trivially, then the degree of the extension $[L : L^G] = |G|$. (Artin; see Dummit and Foote chapter 14, Theorem 19)

**Corollary.** We have $\dim_k R/J \geq |G|$ and if $\dim_k R/J = |G|$, then $R$ is a free $S$-module.

*Proof.* Take basis $e_1, \cdots, e_N$ for $R/J = R/S_+R$. Lift to $\tilde{e_1}, \cdots, \tilde{e_N}$ which span $R$ as an $S$-module. Then $\tilde{e_1}, \cdots, \tilde{e_N}$ span $\operatorname{Frac} R$ as a $\operatorname{Frac} S$ vector space. So $N \geq |G|$. In the equality case, $\tilde{e}_i$ are a basis for $\operatorname{Frac} R$ over $\operatorname{Frac} S$, so there are no nontrivial additive relations between the $\tilde{e}_i$, and thus $R$ is free. ☐

**November 6 – Invariants of Coxeter actions on Laurent polynomial rings.** Let $(W, \Phi)$ be a crystallographic finite Coxeter group, where $\Phi$ is the root system, and $\Lambda = (\mathbb{Z}\Phi^\vee)^\vee$ is the weight lattice. Let $\widehat{R}$ be the ring $\mathbb{R}[\Lambda]$ and let $\widehat{S} = \widehat{R}^W$. Here is the main result today:

**Theorem.** If $\Phi$ spans $V$, then $\widehat{S}$ is a polynomial ring.

As a remark, the reason why we do not impose the spanning condition at the beginning is that we can talk about cases like the action of symmetric group, where we have

$$\mathbb{R}[z_1^\pm, \ldots, z_n^\pm]^{S_n} = \mathbb{R}[e_1, \ldots, e_{n-1}, e_n^\pm].$$

We also note that when $n = \dim V$, where $n = \dim(\mathbb{R}\cdot\Phi)$, then we will have

$$\widehat{R}^W = \mathbb{R}[e_1, \ldots, e_n, s_1^\pm, \ldots, s_{n-r}^\pm].$$

That said, we'll actually only do the detailed proof when $\Phi$ spans.

Let $\alpha_1, \ldots, \alpha_n$ be simple roots, and $\alpha_1^\vee, \ldots, \alpha_n^\vee$ be their corresponding elements in the dual space. We take $w_i$ to be the dual of $\alpha_i^\vee$, in the sense that

$$\langle \alpha_i^\vee, w_j\rangle = \begin{cases} 1 & i = j; \\ 0 & \text{otherwise.}\end{cases}$$

We let $\Lambda = \mathbb{Z} \cdot \{w_1, \dots, w_n\}$, and $D = \mathbb{R}_{\geq 0} \cdot \{w_1, \dots, w_n\}$. Pick $\rho = \sum_{i=1}^n w_i \in \Lambda$. We also write elements of $\mathbb{R}[\Lambda]$ as

$$\sum_{\lambda \in \Lambda} c_\lambda z^\lambda, c_\lambda \in \mathbb{R}.$$

And we say $\sum_{\lambda \in \Lambda} c_\lambda z^\lambda$ is $W$-invariant if $c_\lambda$ is constant on orbits of the action of $W$ on $\Lambda$. From this, we have an $\mathbb{R}$-basis of $\widehat{S}$, given by

$$m_\lambda := \sum_{\delta \in W \cdot \lambda} z^\delta,$$

as $\lambda$ ranges over $\Lambda/W$. As an example, we have $m_{221}(z_1, z_2, z_3) = z_1^2 z_2^2 z_3 + z_1^2 z_2 z_3^2 + z_1 z_2^2 z_3^2$.

We said that $\lambda$ should range over $\Lambda/W$; we know that a set of orbit representatives for that action is $D \cap \Lambda$. So $\{m_\lambda\}_{\lambda \in D \cap \Lambda}$ is a basis of $\widehat{S}$.

Here is our main theorem

**Theorem.** We have

$$\widehat{S} = \mathbb{R}[m_{\omega_1}, \dots, m_{\omega_n}].$$

For $\lambda = \sum c_i \omega_i \in D \cap \Lambda$, we put $e_\lambda = \prod_{i=1}^n m_{\omega_i}^{c_i}$.[2]

**Example.** Let $W = S_2$, and $\Lambda = \mathbb{Z}$. Then we have $\alpha_i = \{2\}$, $\alpha_i^\vee = \{1\}$, and $w_i = \{1\}$. So we have

$$m_0 = 1, m_1 = z + z^{-1}, m_2 = z^2 + z^{-2}, m_3 = z^3 + z^{-3},$$

and

$$\begin{aligned} e_0 &= 1, \\ e_1 &= m_1 = z + z^{-1}, \\ e_2 &= m_1^2 = z^2 + 2 + z^{-2} = m_2 + 2, \\ e_3 &= m_1^3 = z^3 + 3z + 3z^{-1} + z^{-3} = m_3 + 3m_1. \end{aligned}$$

If we want to say $\{e_\lambda\}$ is a upper triangular in $\{m_\lambda\}$, we first have to give a partial order on $\Lambda$. We put

$$\lambda \geq \mu, if \lambda - \mu = Span_{R_{\geq 0}}(\alpha_1, \dots, \alpha_n).$$

We note that for $\lambda \in D \cap \Lambda$, there are only finitely many $\mu \in D \cap \lambda$ with $\mu \leq \lambda$. See Figure 27. Note that we checked on the homework that $D = \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\omega_1, \dots, \omega_n) \subseteq \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\alpha_1, \dots, \alpha_n)$, which is why the shaded acute angle formed by $\omega_1$, $\omega_2$ is inside the obtuse angle formed by $\alpha_1$, $\alpha_2$.

[2] On reflection, to match the standard conventions for symmetric functions, I should call this $e_{\lambda^T}$. But this is confusing, since there is no notion of "transpose" outside type $A$.

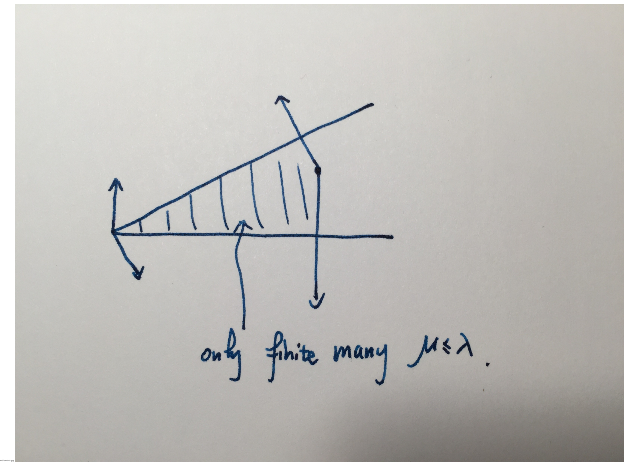



FIGURE 27. There are only finitely many $\mu \in D \cap \Lambda$ which are $\leq \lambda$.

We will show that

$$e_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu, for some K_{\lambda\mu}.$$

If this is true, then for any $\delta \in D \cap \Lambda$, we have

$$Span_{\lambda\leq\delta}(e_\lambda) = Span_{\lambda\leq\delta}(m_\lambda),$$

and both are bases.

Now we reduce the question to showing, if $\mu \in D\cap\Lambda$ such that $m_\mu$ has nonzero coefficient in $e_\lambda$, then $\mu \leq \lambda$. In fact, it is true for all monomials $z^\mu$.

It is enough to show that

$$m_{w_i} = z^{w_i} + \sum_{\delta<w_i} c_\delta z^\delta.$$

This is because every exponent of $m_{w_1}^{c_1} \dots m_{w_n}^{c_n}$ is a sum of $c_1 + \cdots c_n$ terms, of which the first $c_1$ are $\leq w_1$, and the next $c_2$ are $\leq w_2$, etc. For example, we have

$$\begin{aligned} e_{21}(z_1, z_2, z_3) &= (z_1 + z_2 + z_3)^2(z_1z_2 + z_1z_3 + z_2z_3) \\ &= z_1^2 \cdot (z_1z_2) + \cdots \\ &= z^{2w_1+w_2} + \cdots . \end{aligned}$$

Now we only need the following claim:

**Claim.** Let $\gamma \in D$, $w \in W$. Then we have $w \cdot \gamma \leq \gamma$.

*Proof.* We induct on $l(w)$. Let $w = s_{i_1} \dots, s_{i_l}$ be a reduced word. Put $v_k = s_{i_1} \cdots s_{i_k}$, and $\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$. Then $\beta_k \in \Phi^+$. Here is the picture: So we have $v_k\gamma = v_{k-1}\gamma -$ (nonnegative scalar) $\cdot \beta_k$, where $\beta_k \in \Phi^+$. So $v_k\gamma \leq v_{k-1}\gamma \leq \gamma$. ☐

Figure 29 illustrates this claim in $A_2$. The obtuse cone with vertex at the upper right is spanned by $-\alpha_1$ and $-\alpha_2$.

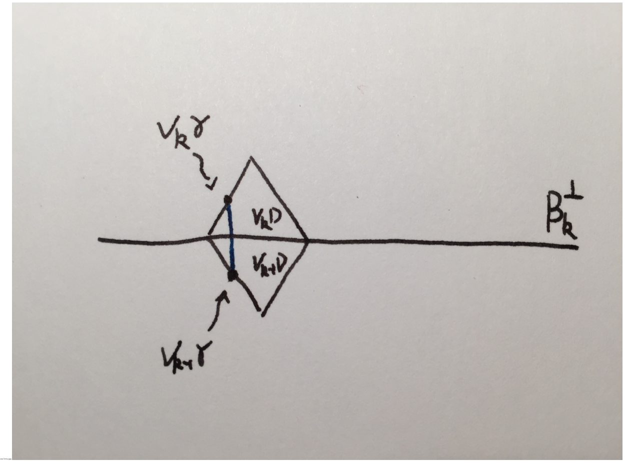



FIGURE 28. $v_k\gamma$ and $v_{k-1}\gamma$

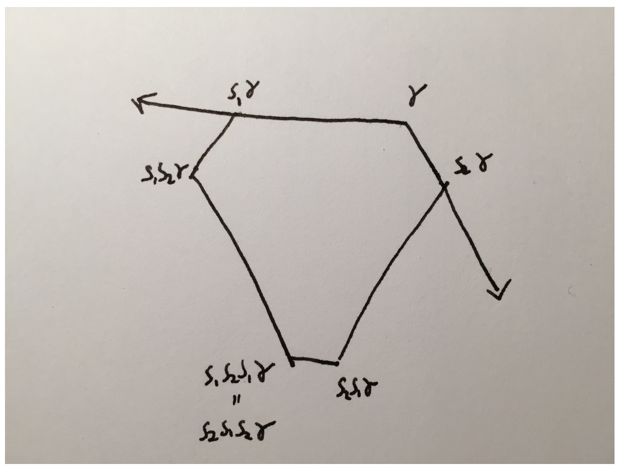



FIGURE 29. Illustration of the final Claim

**November 8 – Anti-symmetric functions.**

**Definition.** Let $W$ be a Coxeter group, $U$ a vector space on which $W$ acts. Then $f \in U$ is called antisymmetric if $s \cdot f = (-1)^{\ell(w)} f$.

Let $R$ be an $\mathbb{R}$-algebra with a $W$ action and $S = R^W$. Then the anti-symmetric elements are an $S$-module.

**Claim.** Let $R = \mathrm{Sym}^\bullet V$ with $V$ the usual representation of $W$. Let $t \in T$ and $f \in S$ obey $f(tx) = -f(x)$. Then $\beta_t$ divides $f$.

*Proof.* Switch to a basis $(x_1, \cdots, x_n)$ of $V$ where $t = \mathrm{diag}(1, 1, \ldots, 1, -1)$. The hypothesis is that $f(x_1, \cdots, x_{n-1}, -x_n) = -f(x_1, \cdots, x_{n-1}x_n)$ so all powers of $x_n$ in $f$ have odd exponent; we conclude that $x_n | f$. □

Now suppose $f$ is anti-symmetric. So every $\beta \in \Phi^+$ divides $f$. So $\Delta := \prod_{\beta \in \Phi^+} \beta | f$.

**Example.** If $f(x_1, \cdots, x_n) \in \mathbb{R}[x_1, \cdots, x_n]$ is anti-symmetric in $(x_1, \cdots, x_n)$ then $\prod_{i<j}(x_i - x_j)|f$.

Notice

$$w(\Delta) = \prod_{\beta\in\Phi^+} w(\beta) = \prod_{\gamma\in w(\Phi^+)} \gamma = (-1)^{\#(w(\phi^+)\cap\phi^-)}\Delta = (-1)^{\ell(w)}\Delta.$$

So $\Delta$ is anti-symmetric. So we see that $f$ is anti-symmetric if and only if $f = \Delta g$ for $g$ symmetric. We conclude:

**Theorem.** The anti-symmetric elements of $R$ are a free rank 1 $S$-module with generator $\Delta$.

We now search for the Laurent polynomial version of this theorem:

Let $\Lambda \subset V$ be a lattice and let $t : \Lambda \to \Lambda$ act by reflection on $V$ so that $V = V_+ \oplus V_-$ with $V_+$ codimension 1, $V_-$ dimension 1. Let $\beta$ be the minimal lattice element in $V_-$ so that $V_- \cap \Lambda = \mathbb{Z}\beta$. Let $\widehat{R} = \mathbb{R}[\Lambda]$.

**Claim.** If $t \cdot f = -f$, then $z^\beta - 1|f$.

This claim is a little messier than the polynomial version, because there is more than one way for a reflection to act on a lattice: $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ acting on $\mathbb{Z}^2$ are not related by an integer change of basis. Nonetheless, it works out; details are left to the reader.

So, let $W$ be a crystallographic Coxeter group and $\Lambda$ a lattice on which it acts. Put $\widehat{R} = \mathbb{R}[\Lambda]$ and suppose $f \in \widehat{R}$ anti-symmetric. Then $z^\beta - 1$ divides $f$ for all $\beta \in \Phi^+$. So $\prod\limits_{\beta\in\Phi^+}(z^\beta - 1)$ divides $f$. This product is not antisymmetric. We now need to tweak things to fix this.

We'd rather work with the product $\widehat{\Delta} := \prod(z^{\frac{\beta}{2}} - z^{-\frac{\beta}{2}})$, which is anti-symmetric. As written, this product is in $\mathbb{R}[\frac{1}{2}\Lambda]$. We now simplify it. We have

$$\widehat{\Delta} = \prod_{\beta\in\Phi^+}(z^{\frac{\beta}{2}} - z^{-\frac{\beta}{2}}) = z^{\sum_{\beta\in\Phi^+}\beta/2}\prod_{\beta\in\Phi^+}(1 - z^{-\beta}) = z^\rho \prod_{\beta\in\Phi^+}(1 - z^{-\beta}).$$

So, if $\rho \in \Lambda$ and $\Phi \subset \Lambda$, then $\widehat{\Delta} \in \widehat{R}$.

So, as before, we deduce:

**Theorem.** If $\Phi$ and $\rho \in \Lambda$, then the anti-symmetric elements of $\widehat{R}$ are a free rank 1 $\widehat{S}$-module with generator $\widehat{\Delta}$.

We take as hypotheses for the rest of the day that $\Phi$ and $\rho$ are in $\Lambda$. This theorem provides a natural new basis for symmetric Laurent polynomials:

For $\mu \in \Lambda$, put $A_\mu = \sum\limits_{w\in W}(-1)^{\ell(w)}z^{w(\mu)}$. So $A_\mu/\widehat{\Delta}$ is symmetric. We have $A_{w(\mu)} = (-1)^{\ell(w)}A_\mu$, so we can restrict our attention to $\mu \in D$. Moreover, if $\mu$ is on the boundary of $D$, then the stabilizer is some nontrivial parabolic subgroup $W_I$ and $A_\mu = 0$. So we can restrict to $\mu \in \Lambda \cap D^0$. As $\mu$ ranges over $D^0 \cap \Lambda$ the $A_\mu$ are a basis of antisymmetric functions.

Note that $\mu \in D_0 \cap \Lambda$ if and only if $\langle \alpha_i^\vee, \mu\rangle \in \mathbb{Z}_{>0}$, if and only if $\langle \alpha_i^\vee, \mu - \rho\rangle \in \mathbb{Z}_{\geq 0}$, which is another way of saying $\mu = \rho + \lambda$ for $\lambda \in D \cap \Lambda$. So a basis for $\hat{S}$ is

$$s_\lambda := \frac{A_{\lambda+\rho}}{\hat{\Delta}}$$

where $\lambda \in D \cap \Lambda$. In type $A$, these are the ***Schur polynomials***.

Recall $\hat{R}^W = \hat{S}$ is functions on $T/W = \mathrm{Conj}(G)$. A good example of a function on conjugacy classes is the trace of a representation. We state the following result, whose proof is fact beyond this course:

**Theorem** (Weyl Character Formula)**.** The $s_\lambda$ are the characters of the finite-dimensional irreducible representations of $G$.

We conclude by discussing the relation of the $s_\lambda$ to the bases $e_\lambda$ and $m_\lambda$ we already posses. Specifically, we will show that

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda,\ \mu\in D\cap\Lambda} K_{\lambda\mu} m_\mu$$

for some constants $K_{\lambda\mu}$. One again, we have ordered $\Lambda$ by $\gamma \leq \delta$ if $\delta - \gamma \in \mathrm{Span}_{\geq 0} \Phi^+$. (In type $A$, the $K_{\lambda\mu}$ are called the ***Kotska numbers***.)

We need to show that, if the monomial $z^\mu$ occurs in $s_\lambda$, then $\mu \leq \lambda$ (and that the coefficient is 1 if $\mu = \lambda$.) We have

$$\begin{aligned} s_\lambda &= \frac{\sum_{w\in W}(-1)^{\ell(w)} z^{w(\lambda+\rho)-\rho}}{\prod_{\beta\in\Phi^+}(1-z^{-\beta})} \\ &= \textstyle\sum_{w\in W}(-1)^{\ell(w)} z^{w(\lambda+\rho)-\rho} \sum_{k_1,k_2,\ldots,k_N=0}^{\infty} z^{-\sum k_i\beta_i} \end{aligned}$$

where $\beta_1, \ldots, \beta_N$ is some enumeration of $\Phi^+$. We have $w(\lambda+\rho) \leq \lambda+\rho$, with equality only for $w = 1$, so $w(\lambda+\rho) - \rho - \sum k_i\beta_i \leq (\lambda+\rho) - \rho = \lambda$.

As a final corollary, we prove

**Theorem** (Weyl Denominator Formula)**.** We have

$$\sum_{w\in W} (-1)^{\ell(w)} z^{w(\rho)} = z^\rho \prod_{\beta\in\Phi^+} (1 - z^{-\beta}).$$

*Proof.* The ratio of the two sides is $s_0$. We showed above that $s_0 = m_0 + (\text{lower order terms})$, but there are no lower order terms, so $s_0 = m_0 = 1$. □

**November 10 – Extensions and Variants of the Weyl Denominator Formula.** At the end of last time, we had $(W, \Phi)$, a finite crystallographic Coxeter group and root system. We also had $\Lambda$, a lattice on which $W$ acts, with $\Phi \subset \Lambda$, and $\rho = \frac{1}{2} \sum_{\beta\in\Phi^+} \beta \in \Lambda$. Lastly, we had

$$\widehat{\Delta} = \prod_{\beta\in\Phi^+} (z^{\frac{\beta}{2}} - z^{\frac{-\beta}{2}}) = z^\rho \prod_{\beta\in\Phi^+} (1 - z^{-\beta}).$$

For any $\lambda \in \Lambda \cap D$,

$$\frac{\sum_{w\in W} (-1)^{\ell(w)} z^{w(\lambda+\rho)}}{\widehat{\Delta}}$$

is symmetric. What if $\lambda = 0$? Then, we get $\frac{\sum_{w\in W}(-1)^{\ell(w)} z^{w(\rho)}}{\widehat{\Delta}} = 1$. So we have the Weyl denominator formula:

$$\sum_{w\in W} (-1)^{\ell(w)} z^{w(\rho)} = \prod_{\beta\in\Phi^+} (z^{\frac{\beta}{2}} - z^{\frac{-\beta}{2}}) = z^\rho \prod_{\beta\in\Phi^+} (1 - z^{-\beta}).$$

**Example.** Take $S_n$ acting on $\mathbb{Z}^n/\mathbb{Z}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$. The simple roots are $\{e_1 - e_2, \dots, e_{n-1} - e_n\}$, giving us $\rho = (n-1, n-2, \dots, 2, 1, 0)$. Our formula then gives us:

$$\sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} z_{\sigma(1)}^{n-1} z_{\sigma(2)}^{n-2} \dots z_{\sigma(n)}^{0} = z_1^{n-1} z_2^{n-2} \dots z_{n-1} \prod_{i<j} (1 - \frac{z_j}{z_i}).$$

The left side is

$$\det \begin{bmatrix} z_1^{n-1} & z_2^{n-2} & \dots & z_n^{n-1} \\ \vdots & \ddots & & \vdots \\ z_1 & z_2 & \dots & z_n \\ 1 & 1 & \dots & 1 \end{bmatrix} = \prod_{\mathrm{i<j}} (\mathrm{z_i} - \mathrm{z_j}),$$

this is Vandermonde's determinant identity.

Types $B$, $C$ and $D$ each can also be rewritten as determinants; see the course website.

Put $\widehat{\Delta}_q := z^\rho \prod_{\beta\in\Phi^+} (1 - qz^{-\beta})$. We will now play games with $\widehat{\Delta}_q$.

**Example.** In our example from before, but now with $S_3$, we get

$$z_1^2 z_2 (1 - q\frac{z_2}{z_1})(1 - q\frac{z_3}{z_1})(1 - q\frac{z_3}{z_2}) = z_1^2 z_2 - q z_1 z_2^2 - q z_1^2 z_3 + q^2 z_2^2 z_3 + q^2 z_1 z_3^2 - q^3 z_2 z_3^2 + (q^2 - q) z_1 z_2 z_3.$$

In our previous proof, we saw two things:

- Every $z^\mu$ that occurs with non-zero coefficient in $\widehat{\Delta}_q$ has $\mu \le \rho$ (we get to them by taking $\rho$ and subtracting positive roots).
- The set of $\mu$ which occur is $W$-invariant. This latter is probably easiest to see if we write $\widehat{\Delta}_q = \prod_{\beta\in\Phi^+} (z^{\beta/2} - qz^{-\beta/2})$: Each exponent is of the form $(1/2)\sum_{\beta\in X} \beta$ where $X$ contains exactly one of each $\pm\beta$ pair.

Now, we want to antisymmetrize. So $\sum_{w\in W} (-1)^{\ell(w)} w \cdot \widehat{\Delta}_q$ is a multiple of $\widehat{\Delta}$ and, looking at the coefficient of $z^\rho$, we have

$$\sum_{w\in W} (-1)^{\ell(w)} w \cdot \widehat{\Delta}_q = (\sum_{w\in W} q^{\ell(w)}) \widehat{\Delta}.$$

We remove the monomials lying on the supporting hyperplanes.

**Example.** Again in our $S_3$ example, we see $(q^2 - q) z_1 z_2 z_3$ gets taken out three times. We have

$$\sum_{\sigma\in S_3} (-1)^{\ell(\sigma)} (z_{\sigma(1)} - q z_{\sigma(2)})(z_{\sigma(1)} - q z_{\sigma(3)})(z_{\sigma(2)} - q z_{\sigma(3)}) = (1 + 2q + 2q^2 + q^3)(z_1 - z_2)(z_1 - z_3)(z_2 - z_3).$$

Rearranging, we have

$$\sum_{w\in W} q^{\ell(w)} = \sum_{w\in W} \frac{\prod_{\beta\in\Phi^+} (1 - qz^{-w(\beta)})}{\prod_{\beta\in\Phi^+} (1 - z^{-w(\beta)})}.$$

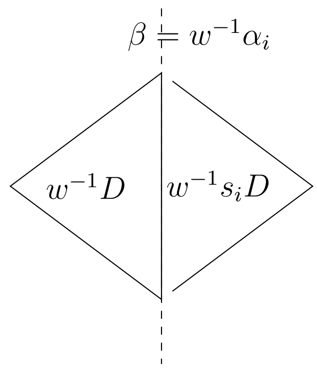
**Example.** In $S_2$, we check:

$$\frac{1 - qz^{-1}}{1 - z^{-1}} + \frac{1 - qz}{1 - z} = 1 + q.$$

We get a nice specialization of the $z$'s by sending $z^{\alpha_i} \mapsto q^{-1}$. So if $\beta = \sum c_i\alpha_i$, $z^\beta \mapsto q^{-\sum c_i}$. The term where $w$ is the identity is:

$$\frac{\prod_{\beta\in\Phi^+}(1-qz^{-\beta})}{\prod_{\beta\in\Phi^+}(1-z^{-\beta})} = \prod_{\beta\in\Phi^+}\frac{1-q^{\mathrm{ht}(\beta)+1}}{1-q^{\mathrm{ht}(\beta)}}.$$

If $w$ is not the identity, the $w$ has some left descent, $s_i$. This gives us the picture:



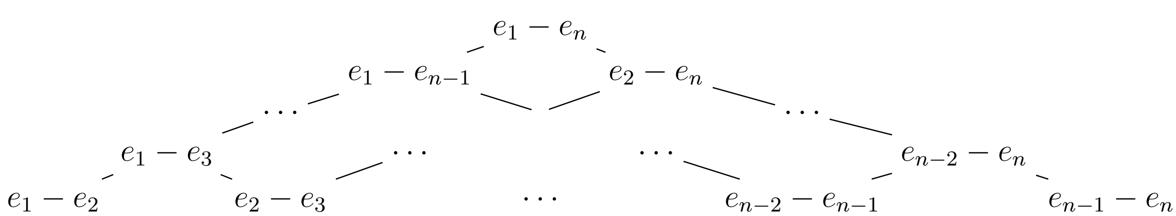

with $w\beta = -\alpha_i$, where $\beta^\perp$ separates $w^{-1}D$ and $w^{-1}s_iD$, and $\beta$ is in $\Phi^+$. This gives us that $\prod_{\beta\in\Phi^+}(1-qz^{-\mu(\beta)})$ specializes to 0. So only one term survives the specialization, we deduce

$$\sum_{w\in W} q^{\ell(w)} = \prod_{\beta\in\Phi^+}\frac{1-q^{\mathrm{ht}(\beta)+1}}{1-q^{\mathrm{ht}(\beta)}}.$$

The height function is on the root poset. For a more usable form, we note that if there are $h_i$ elements of $\Phi^+$ at height $i$, then the right hand side is equal to

$$\frac{\prod_{i=2}^{\infty}(1-q^i)^{h_{i-1}-h_i}}{(1-q)^n}.$$

**Example.** In $A_{n-1}$, our root poset looks like:




So $(h_1, h_2, \dots, h_{n-1}) = (n-1, n-2, \dots, 1)$, giving us

$$\sum_{w\in S_n} q^{\ell(w)} = \frac{(1-q^2)(1-q^3)\dots(1-q^n)}{(1-q)^{n-1}}.$$

In particular, sending $q$ to 1,

$$|W| = \prod_{\beta\in\Phi^+}\frac{\mathrm{ht}(\beta)+1}{\mathrm{ht}(\beta)}.$$

**November 13 – Regular Rings.** Let $R$ be a Noetherian commutative ring, $\mathfrak{m}$ a maximal ideal, and $k := R/\mathfrak{m}$ a field. Note that $V = \mathfrak{m}/\mathfrak{m}^2$ is a $k$-vector space called the ***Zariski cotangent space***. Since $R$ is Noetherian, $\mathfrak{m}$ is finitely generated, whence $V$ is finite dimensional. For any $d \geq 0$, we have a natural map of $k$-vector spaces

$$\mathrm{Sym}^d V \to \mathfrak{m}^d/\mathfrak{m}^{d+1}.$$

By Nakayama's lemma, this map is surjective. We define $R$ to be ***regular at*** $\mathfrak{m}$ if this map is an isomorphism for all $d$. We call $R$ a ***regular ring*** if $R$ is regular at every maximal ideal.

Some Examples:

(1) First, we work with polynomial rings over a fixed field $k$.
  (a) $R = k[x]$ is regular at $\mathfrak{m} = \langle x - a \rangle$ for any $a \in k$.
  (b) $R = k[x_1, \dots, x_n]$ is regular at $\mathfrak{m} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$ for any $a_1, \dots, a_n \in k$.
  (c) $R = k[x^{\pm 1}]$ is regular at $\mathfrak{m} = \langle x - 1 \rangle$. To see this, notice that

$$S := k[x^{\pm 1}]/\langle x-1 \rangle^N \cong k[x]/\langle x-1 \rangle^N$$

  because, in $S$, we have $x^{-1} = \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^{N-1}$. So once we are quotienting by powers of $\mathfrak{m}$ any way, it doesn't matter that we were inverting $x$.

(2) $R = \mathbb{Z}$ is regular at $\mathfrak{m} = \langle p \rangle$ for any prime $p$.

(3) $R = \mathbb{Z}[x]$ is regular at $\mathfrak{m} = \langle p, x \rangle$ for any prime $p$. We have $V = \mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{F}_p^2$.

By contrast, the ring $R = k[x,y]/\langle xy \rangle$ is not regular at $\mathfrak{m} = \langle x, y \rangle$. We have

$$\mathfrak{m}/\mathfrak{m}^2 = \frac{\langle x, y \rangle}{\langle x^2, y^2 \rangle} \cong k^2 \text{ and } \mathrm{Sym}^d(\mathfrak{m}/\mathfrak{m}^2) \cong k^{d+1} \text{ for all } d.$$

But for $d = 2$ we have

$$\mathfrak{m}^2/\mathfrak{m}^3 = \frac{\langle x^2, y^2 \rangle}{\langle x^3, x^2y, xy^2, x^3, xy \rangle} \cong k^2.$$

Roughly speaking, regular rings have no "higher degree relations" like $xy = 0$ in this example, but they define this without needing to introduce a notion of degree.

Suppose that we have an embedding $k \hookrightarrow R$ such that $k \hookrightarrow R/m \twoheadrightarrow k$ is an isomorphism. Let $e_1, \dots, e_N$ be a basis of $V$ over $k$, and choose lifts $\widetilde{e_1}, \dots, \widetilde{e_N}$ to $m$. We get a map

$$k[x_1, \dots, x_N] \to R, \quad x_j \mapsto \widetilde{e_j}.$$

This in turn descends to

$$k[x_1, \dots, x_N]/\langle x_1, \dots, x_N \rangle^{d+1} \to R/m^{d+1}$$

for each $d \geq 0$.

**Claim.** $R$ is regular at $m$ if and only if the latter map is an isomorphism for all $d$.

*Proof.* The left hand side is filtered by $\langle x_1, \dots, x_N \rangle^e$ while the right hand side is filtered by $m^e$. The map preserves the filtration. In high tech language, the map between filtered vector spaces will be an isomorphism if and only if the map on associated graded vector spaces is an isomorphism. In low tech language, the map is given by a lower block-triangular matrix, so it is invertible if and only if the diagonal blocks are invertible. □

Now suppose $R$ is graded, and set $\mathfrak{m} = R_+ := \bigoplus_{i>0} R_i$. Suppose $R_0$ is a field $k$.

**Claim.** $R$ is regular at $\mathfrak{m} = R_+$ if and only if $R$ is a polynomial ring with homogeneous generators.

*Proof.* The vector space $V = \mathfrak{m}/\mathfrak{m}^2$ inherits the induced grading. Choose a homogeneous basis $e_1, \ldots, e_N$ of $V$, and take homogeneous lifts $\widetilde{e_1}, \ldots, \widetilde{e_N}$ to $R_+$. Let $d_i := \deg e_i$. We have a natural graded map

$$k[x_1, \ldots, x_N] \to R$$

where the left hand side is graded with $\deg x_i = d_i$.

We now show this map is an isomorphism. It is enough to check that it is an isomorphism in each degree $D$. Take any $d > D$: our hypothesis is that

$$k[x_1, \ldots, x_N]/\langle x_1, \ldots, x_N \rangle^{d+1} \to R/R_+^{d+1}$$

is an isomorphism. We take the degree $D$ part of this map. Since $d > D$, the ideals $\langle x_1, \ldots, x_N \rangle^{d+1}$ and $R_+^{d+1}$ are 0 in degree $D$. We deduce that $k[x_1, \ldots, x_N]_D \to R_D$ is an isomorphism, as desired. □

Finally, we note that if $R$ is a graded ring, polynomial with generators in degrees $d_1, \ldots, d_n$, then the degrees are uniquely determined, since we have the formal series

$$\sum_{i=0}^{\infty} (\dim R_i) t^i = \frac{1}{\prod_{j=1}^{n} (1 - t^{d_j})}.$$

**November 15 – Rings of Coxeter invariants are polynomial, first proof.** Let $(W, \Phi)$ be a crystallographic Coxeter group. Let $\Phi$ span $V$ and let $\Lambda \subset V$ be the weight lattice $(\mathbb{Z}\Phi^\vee)^\vee$. Let $R = \mathrm{Sym}^\bullet(V)$ and let $\widehat{R} = \mathbb{R}[\Lambda]$. Let $S = R^W$ and $\widehat{S} = \widehat{R}^W$. We saw on November 6 that $\widehat{S}$ is a Laurent polynomial ring. Our goal today is to deduce that $S$ is also a Laurent polynomial ring. This is part of:

**Theorem** (Chevalley-Shephard-Todd)**.** Let $K$ be a field of characteristic zero and let $W$ be a finite subgroup of $GL_n(K)$. Let $R = K[x_1, \ldots, x_n]$ and $S = R^W$. Then $S$ is a polynomial ring if and only if $W$ is generated by elements which fix $n-1$ dimensional subspaces of $K^n$.

The rough idea of our proof is to map $\widehat{R}$ to $R$ by

$$z^\beta \mapsto \sum_{n=0}^{\infty} \frac{\beta^n}{n!}.$$

So we "plug in $e$ for $z$". Infinite sums don't make sense in $R$, but this formula does define a map $\phi : \widehat{R} \to R/R_+^{d+1}$ for any $d$.

Geometrically, $\phi$ comes from the exponential map from the Lie algebra $\mathfrak{t}$ to the torus $T$. This map only exists in the analytic category (which is why we don't have an algebraic map $\widehat{R} \to R$) but the map $\phi$ is an algebraic shadow of it.

We put $\mathfrak{m} = R_+ \subset R$ and $\widehat{\mathfrak{m}} = \langle z^\beta - 1 \rangle \subset \widehat{R}$.

**Lemma.** The map $\phi$ descends to an isomorphism of $\mathbb{R}$-algebras $\widehat{R}/\widehat{\mathfrak{m}}^{d+1} \to R/\mathfrak{m}^{d+1}$. This isomorphism takes $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^{d+1}$ to $\mathfrak{m}/\mathfrak{m}^{d+1}$ and commutes with the $W$ action.

*Proof.* We first check that $\phi$ is a map of rings. We must show

$$\phi(z^{\beta+\gamma}) = \phi(z^\beta)\phi(z^\gamma)$$

or, in other words

$$\sum_{n=0}^{\infty} \frac{(\beta+\gamma)^n}{n!} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!}.$$

This follows from the binomial theorem: $(\beta+\gamma)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \beta^j \gamma^{n-j}$.

Next, we note that $\widehat{\mathfrak{m}}$ is taken into $\mathfrak{m}$, since $\phi(z^\beta - 1) = \sum_{n=1}^{\infty} \frac{\beta^n}{n!} \in \mathfrak{m}$. This makes it clear that the map descends to $\widehat{R}/\widehat{\mathfrak{m}}^{d+1} \to R/\mathfrak{m}^{d+1}$. Since $R$ and $\widehat{R}$ are regular at $\mathfrak{m}$ and $\widehat{\mathfrak{m}}$ respectively, to check that the map is an isomorphism, we just need to check that it is an isomorphism on the Zariski cotangent spaces $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \to \mathfrak{m}/\mathfrak{m}^2$. We have $\mathfrak{m} = \bigoplus_{i \geq 1} R_i$ and $\mathfrak{m}^2 = \bigoplus_{i \geq 2} R_i$ so $\mathfrak{m}/\mathfrak{m}^2 = R_1 = V$. Meanwhile, in $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$, we have $(z^\beta - 1) + (z^\gamma - 1) \equiv z^{\beta+\gamma} - 1$, since the difference $z^{\beta+\gamma} - z^\beta - z^\gamma + 1 = (z^\beta - 1)(z^\gamma - 1)$ lies in $\mathfrak{m}^2$. So $z^\beta - 1 \mapsto \beta$ provides a map $\widehat{fm}/\widehat{fm}^2 \to V$, which is easily seen to be an isomorphism.

Finally, we note that $\phi$ commutes with the action of $W$:

$$\phi(z^{w\beta}) = \sum_{n=0}^{\infty} \frac{(w\beta)^n}{n!} = w\left(\sum_{n=0}^{\infty} \frac{\beta^n}{n!}\right) = w\phi(z^\beta). \qquad \square$$

Put $\mathfrak{n} = S \cap \mathfrak{m}$ and $\widehat{\mathfrak{n}} = \widehat{S} \cap \widehat{\mathfrak{m}}$.

We know that $\widehat{S}$ is the polynomial ring in $e_i := \sum_{\beta \in W\omega_i} z^\beta$. We claim that $\widehat{\mathfrak{n}} = \langle e_i - |W\omega_i| \rangle$. Clearly, $e_i - |W\omega_i| = \sum_{\beta \in W\omega_i} (z^\beta - 1)$ is in $\widehat{\mathfrak{n}}$. But $\widehat{S}/\langle e_i - |W\omega_i| \rangle \cong \mathbb{R}$, so $\langle e_i - |W\omega_i| \rangle$ is already maximal so $\widehat{\mathfrak{m}} \cap \widehat{S}$ couldn't be larger than this. We see that $\widehat{S}$ is regular at $\widehat{fm}$.

For any positive integer $D$, we have

$$(R/\mathfrak{m}^D)^W \cong (\widehat{R}/\widehat{\mathfrak{m}}^D)^W$$

since $R/\mathfrak{m}^D$ and $\widehat{R}/\widehat{\mathfrak{m}}^D$ are isomorphic as rings with $W$-action. We claim that $(R/\mathfrak{m}^D)^W \cong S/(\mathfrak{m}^D \cap S)$ and likewise for the Laurent versions. Clearly, we have a map $S = R^W \to (R/\mathfrak{m}^D)^W$ and its kernel, by definition, is $\mathfrak{m}^D \cap S$. So $S/(\mathfrak{m}^D \cap S)$ injects into $(R/\mathfrak{m}^D)^W$ and the issue is to prove surjectivity. Let $f \in (R/\mathfrak{m}^D)^W$ and lift $f$ to $\tilde{f} \in R$. Then $\tilde{f}$ may not be $W$ invariant, but $\frac{1}{|W|} \sum_{w \in W} \tilde{f}^w$ is and has the same image in $(R/\mathfrak{m}^D)^W$. So we can find a $W$-invariant lift of $f$. Then this lift is an element of $S$ which maps to $f$.

We deduce

$$\frac{S}{\mathfrak{m}^D \cap S} \cong \frac{\widehat{S}}{\widehat{\mathfrak{m}}^D \cap \widehat{S}}.$$

Also, this morphism carries the image of $\widehat{\mathfrak{n}}$ to the image of $\mathfrak{n}$. Thus,

$$\frac{S}{\mathfrak{n}^E + (\mathfrak{m}^D \cap S)} \cong \frac{\widehat{S}}{\widehat{\mathfrak{n}}^E + (\widehat{\mathfrak{m}}^D \cap \widehat{S})}$$

for any positive integer $E$. But (homework!) $\mathfrak{m}^{E|W|} \cap S \subseteq \mathfrak{n}^E$. So taking $D \geq E|W|$ we deduce

$$S/\mathfrak{n}^E \cong \widehat{S}/\widehat{\mathfrak{n}}^E$$

for any $E$. We know $\widehat{S}$ is regular at $\widehat{\mathfrak{m}}$, so this implies that $S$ is regular at $\mathfrak{m}$. By the previous day's lemma, this shows $S$ is polynomial. **QED**

**November 17 – Molien's formula and consequences.** Recall from homework, if $G$ is a finite group, $\rho : G \to GL(V)$ is a representation, where $V$ is a finite dimensional vector space over $\mathbb{R}$, we have

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} \rho(g).$$

*Proof.* Put $\pi = \frac{1}{|G|} \sum \rho(g)$. So $\pi$ is an endomorphism of $V$ satisfying $\pi^2 = \pi$, and we have $\operatorname{Im} \pi = V^G$. So $\dim V^G = \operatorname{Tr} \pi = \frac{1}{|G|} \sum \operatorname{Tr} \rho(g)$. □

This generalizes Burnside's Theorem. To see the connection, let $G$ act on a finite set $X$ and let $V = \mathbb{R}^X$. Then $V^G$ are functions constant on orbits, so $dimV^G = \#(X/G)$, and $\operatorname{Tr} g = \# \operatorname{Fix}(g)$. So we deduce Burnside's theore: $\#(X/G) = \frac{1}{|G|} \sum_{g \in G} \# \operatorname{Fix}(g)$.

Let $R = \operatorname{Sym}^\bullet V$ and let $S = R^G$. Applying the previous result degree by degree, we get:

$$\sum_{i=0}^{\infty} (\dim S_i) t^i = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{\infty} t^i \operatorname{Tr}(g : \operatorname{Sym}^i V \to \operatorname{Sym}^i V)$$

Let $g \in G$ act on $V$ with eigenvalues $\lambda_1(g), \ldots, \lambda_n(g)$. Then we have:

$$\sum_{i=0}^{\infty} (\dim S_i) t^i = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\Pi_{i=1}^n (1 - t\lambda_i(g))} = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - t\rho(g))}.$$

This formula is known as Molien's Formula.

Now suppose $S$ is a polynomial ring generated by $f_1, \ldots, f_n$ with degrees $d_1, \ldots, d_n$. So

$$\frac{1}{\Pi_{i=1}^n (1 - t^{d_i})} = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\Pi_{i=1}^n (1 - \lambda_i(g) t)}.$$

We rewrite

$$|G| = \sum_{g \in G} \frac{\Pi_{i=1}^n (1 - t^{d_i})}{\Pi_{j=1}^n (1 - \lambda_j(g) t)}.$$

Let $t \to 1$. All terms except $g = \operatorname{Id}$ go to 0. So we obtain

$$|G| = \lim_{t \to 1} \prod_{i=1}^n \frac{1 - t^{d_i}}{1 - t} = \prod_{i=1}^n d_i$$

Lets expand around $t = 1$ up to $O((t-1)^2)$. We only need to keep $Id$ and those $g$ with eigenvalues $(1, 1, \ldots, \zeta)$. Since we are over $\mathbb{R}$, $\zeta$ must be -1, and $g$ must be a reflection. Write $T$ to be the set of all reflections in $G$:

$$|G| = \prod_{i=1}^n \left( d_i - \frac{d_i(d_i - 1)}{2}(1 - t) + O(1 - t)^2 \right) + |T| \frac{(1 - t)}{2} \prod d_i + O(1 - t)^2$$

Factoring $\prod d_i = |G|$ out of everything, we obtain

$$1 = \prod_{i=1}^n \left( 1 - \frac{(d_i - 1)}{2}(1 - t) + O(1 - t)^2 \right) + |T| \frac{(1 - t)}{2} + O(1 - t)^2$$

So

$$|T| = \sum_{i=1}^n (d_i - 1).$$

These two results are special cases of a Theorem of Shephard and Todd (1954, *Canad. J. Math*), given a better proof by Solomon (1963, *Nagoya Math. J.*). Namely:

$$\prod_{i=1}^{n}(q+d_i-1)=\sum_{w\in W} q^{\dim V^w}.$$

The $q^n$ term has coefficient 1, meaning that only Id acts trivially. The coefficient of $q^{n-1}$ gives $|T|=\sum(d_i-1)$ and putting $q=1$ gives $\prod d_i=|W|$. We'll prove this next week.

We next discuss the topic of, if we find invariants in $S$, how can we know they generate $S$? Let $W$ be a finite coxeter group, so we have $R^W=S$ is a polynomial ring, suppose we find algebraically independent $h_1,\ \ldots,\ h_n\in S$ with degrees $e_i$ obeying either $\sum(e_i-1)=|T|$ or $\prod e_i=|W|$, I claim $S=\mathbb{R}[h_1,\ldots,h_n]$.

Let $S=\mathbb{R}[f_1,\ldots,f_n]$ and let the degree of $f_i$ be $d_i$. I claim we can reorder the $h_i$ such that $e_i\geq d_i$. We can write $h_j=P_j(f_1,\ldots,f_n)$. Define bipartite graph with vertices $f_1,\ldots,$ $f_n$, $h_1,\ \ldots,\ h_n$, and edges $f_i\to h_j$ if $f_i$ occurs in $P_j$. If we have an edge $f_i\to h_j$, then $degh_j\geq degf_i$. We need to know Hall's Marriage Theorem:

**Theorem.** Exactly one of the following occurs:

(1) We can reorder the $h_i$, such that these edges $f_i\to h_i$, for any i

(2) There exists a subset $Q\subset[n]$, such that $\#\{i:\exists j\in Q, f_i\to h_j\}<\#(Q)$

Assume for the sake of contradiction we're in Case (2). Qo $\{h_j : j\in Q\}$ are polynomials in $\{f_i : i\in P\}$ for some $|P|<|Q|$. But then $\{h_j : j\in Q\}$ is not algebraically independent, so a contradiction. So we are in Case (1) and, after reordering, we have $e_i\geq d_i$. But $\prod e_i=\prod d_i=|G|$, so $e_i=d_i$ (and similarly if $\sum(e_i-1)=|T|=\sum(d_i-1)$.

Let $A=\mathbb{R}[h_i,\ldots,h_n]\subset S=\mathbb{R}[f_i,\ldots,f_n]$. Then we have $\sum(\dim A_i)t^i=\prod\frac{1}{(1-t_i^d)}=\sum(\dim S_i)t^i$. So $A_i=S_i$, as claimed.

We can now prove the reverse direction of the Chevalley-Shephard-Todd theorem:

**Theorem.** Let $G\subset GL_n(\mathbb{R})$ be a finite group. and $R^G=\mathbb{R}[h_1,\ldots,f_n]$. Then $G$ is a reflection group.

*Proof.* Let $T$ be the set of reflections in $G$. Let $W=\langle T\rangle\subset G$. Then $R^W=\mathbb{R}[f_i,\ldots,f_n]$. So $\sum(\deg f_i-1)=\sum(\deg h_i-1)=|T|$, so $\deg f_i=\deg h_i$ after reordering. So $|W|=\prod(\deg f_i)=\prod(\deg h_i)=|G|$. We deduce that $W=G$. □

**November 20 – The ring of invariant differentials.** This lecture and the next closely follow Solomon, "Invariants of finite reflection groups", 1963.

As usual, let $W$ be a finite Coxeter group and let $W$ act on $V$. Let $R=\mathrm{Sym}^\bullet V$ (functions on $V^\vee$). Let $S=R^W$.

In addition to functions on vector spaces, we can also have differential forms on vector spaces. Let $E$ be the non-commutative algebra of differential forms on $V^\vee$ with polynomial coefficients. So if $R=\mathbb{R}[x_1,...,x_n]$, then $E=\mathbb{R}[x_1,...,x_n]\langle dx_1,...,dx_n\rangle$. Our goal today is understand $E^W$.

We proved $S$ is a polynomial ring. Let $S=\mathbb{R}[f_1,...,f_n]$. Today's main result is

**Theorem.**

$$E^W=\mathbb{R}[f_1,\ldots,f_n]\langle df_1,\ldots,df_n\rangle.$$

Obviously, everything on the right hand side is invariant. It remains to show that this gives us all of $E^W$.

Recall that, given vector spaces $\mathbb{R}^m, \mathbb{R}^n$, and a $C^\infty$ map $\phi : \mathbb{R}^m \to \mathbb{R}^n$, we get a $m \times n$ matrix $D\phi = \left[\frac{\partial \phi_i}{\partial x_j}\right]$ which obeys the chain rule: $[D(\phi \circ \psi)]_x = [D\phi]_{\psi(x)}[D\psi]_x$.

**Lemma.** Let $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$. The determinant $\det Df$ is **not** identically 0 if and only if $f_1, \ldots, f_n$ are algebraically independent. This holds in any field of characteristic zero.

*Proof.* The easy direction is if $f_1, \ldots, f_n$ are algebraically independent, say $h(f_1, \ldots, f_n) = 0$. Then $Dh \circ Df = 0$ (this is a $1 \times n$ matrix times an $n \times n$ matrix. Taking $h$ of minimal degree with $h(f_1, \ldots, f_n) = 0$, not all entries of $Dh$ are 0. (Note that the entries of $Dh$ are $\frac{\partial h}{\partial y_i}(f_1, \ldots, f_n)$.) So $Df$ has a kernel and $\det Df = 0$.

Now for the reverse direction. Suppose $f_1, \ldots, f_n$ are algebraically independent. We need to show that $\det \left[\frac{\partial f_i}{\partial x_j}\right] \neq 0$. Any $n+1$ polynomials in $n$ variables are algebraically dependent. So there exists some $g_j \in \mathbb{R}[y_1, ..., y_n, z]$ such that $g_j(f_1, ..., f_n, x_j) = 0$. Choose $g_j$ to have minimal degree with respect to $z$. Since $f_1 \ldots, f_n$ are algebraically independent, this degree is greater than 0. We are in characteristic 0 so the polynomials $\frac{\partial g_j}{\partial z} \neq 0$ are nonzero in $\mathbb{R}(y_1, \ldots, y_n, z)$. Since we chose $g_i$ of minimal degree, we also have $\frac{\partial g_i}{\partial z}(f_1, f_2, \ldots, f_n, x_j) \neq 0$ as a polynomial in $x_1, \ldots, x_n$.

Take some $\bar{x}_1, \ldots, \bar{x}_n \in \mathbb{R}^n$ where all the polynomials $\frac{\partial g_i}{\partial z}(f_1, f_2, \ldots, f_n, x_j)$ are nonzero. By the Implicit Function Theorem, there exist functions $\zeta_1, \ldots, \zeta_n$ defined near $(f_1(\bar{x}), ..., f_n(\bar{x}))$ with $\zeta_j(f(\bar{x})) = \bar{x}$ and $g_j(f_1(x), ..., f_n(x), \zeta_j(x)) = 0$. So $(\zeta_1, \ldots, \zeta_n)$ locally invert $(f_1, \ldots, f_n)$. It follows that

$$\left[\frac{\partial \zeta_j}{\partial y_i}\right]\left[\frac{\partial f_i}{\partial x_j}\right] = Id_n.$$

See the handout/website for a purely algebraic proof (taken from Humphrey's.) ☐

Now let $S = \mathbb{R}[f_1, \ldots, f_n]$. So we know $\det \left[\frac{\partial f_i}{\partial x_j}\right] = \det Df \neq 0$. We claim that $\det Df$ will be anti-symmetric. To see this, for any $w \in W$, note that we have $f \circ w = f$, so $(Df)_{w(x)} \cdot w = Df_x$ and $\det(Df)_{w(x)}(-1)^\ell(w) = (Df)_x$. But, unpacking the notation, $(Df)_{w(x)}$ is what we mean by $w \cdot Df$, so $w \cdot Df = (-1)^{\ell(w)} Df$.

Now note that $\deg \left[\frac{\partial f_i}{\partial x_j}\right] = (\deg f_i) - 1 = d_i - 1$, so $\deg \det \left[\frac{\partial f_i}{\partial x_j}\right] = \sum(d_i - 1) = |T| = \deg \Delta$. Since $Df$ is antisymmetric, it is divisible by $\Delta$, and we just showed it has the same degree as $\Delta$, so

$$Df = c\Delta$$

for some nonzero scalar $c$. We can rescale the $f$'s to make $c = 1$.

We want to understand $E^W$. As a warm up, let's see when $g \cdot dx_1 \wedge \cdots \wedge dx_n$ will be invariant. We have

$$w \cdot (dx_1 \wedge \cdots \wedge dx_n) = (-1)^{\ell(w)} dx_1 \wedge \cdots \wedge dx_n$$

So $gdx_1 \wedge \cdots \wedge dx_n)$ is $W$-invariant if and only if $g$ is anti-symmetric. In this case, we have $g = h\Delta$ for $h \in S$. Then

$$g\ dx_1 \wedge \cdots \wedge dx_n = h\ \Delta dx_1 \wedge \cdots \wedge dx_n = h\ df_1 \wedge \cdots \wedge df_n.$$

So, invariant $n$-forms are $S \cdot (df_1 \wedge \cdots \wedge df_n)$.

We now consider arbitrary $p$-forms with $p \neq n$. Let $E = \oplus E^p$ where $E^p$ are $p$-forms. Let $K = \operatorname{Frac} R$. Let $L = \operatorname{Frac} S = K^W$. Then $E^p \otimes_R K$ is a rank $\binom{n}{p}$ $K$-vector space with obvious basis $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ where $1 \leq i_1 < i_2 < ... < i_p \leq n$.

**Claim**: $df_{i_1} \wedge \cdots \wedge df_{i_p}$ is also a basis.

*Proof.* Since this set has the same number of elements as the basis, it is enough to check that the elements are linearly independent. Suppose $\sum a_{i_1 \cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p} = 0$ for some $a_{i_1 \cdots i_p} \in K$. We need to show that $a_{i_1 \cdots i_p} = 0$ for all coefficients.

$$\left(\sum a_{i_1 \cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p}\right) \wedge (df_{p+1} \wedge \cdots \wedge df_n) = 0$$
$$a_{1 \cdots p} df_1 \wedge \cdots \wedge df_n = 0$$

Since $df_1 \wedge \cdots \wedge df_n \neq 0$, it follows that $a_{1 \cdots p} = 0$, and similar arguments apply to all the coefficients $a_{i_1 \cdots i_p}$. So any $\omega \in E^p$ can be uniquely written as $\sum a_{i_1 \cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p}$ with $a_{i_1 \cdots i_p} \in K$. This is $W$-invariant if and only if $a_{i_1 \cdots i_p} \in K^W = L$. Since $\omega$ is in $E$, so is

$$\omega \wedge df_{p+1} \wedge \cdots \wedge df_n = a_{1 \cdots p} df_1 \wedge \cdots \wedge df_n = a_{1 \cdots p} c \Delta dx_1 \wedge \cdots \wedge dx_n$$

So $a_{1 \cdots p} \Delta \in R$. More generally, $a_{i_1 \cdots i_p} \Delta \in R$ and $a_{i_1 \cdots i_p} \in \operatorname{Frac} S$.

So $a_{i_1 \cdots i_p} \Delta$ is anti-symmetric. So $a_{i_1 \cdots i_p} \Delta = b_{i_1 \cdots i_p} \Delta$ for $b_{i_1 \cdots i_p} \Delta \in S$. So $a_{i_1 \cdots i_p} = b_{i_1 \cdots i_p} \in S$. So $E^W = \mathbb{R}[f_1, \cdots, f_n]\langle df_1, \cdots, df_n \rangle$. □

**November 22 – The formula of Shephard and Todd.** Recall where we were last time. Let $R = \operatorname{Sym}^\bullet(V)$ and let $E$ be the ring of polynomial differential forms on $V^\vee$. Let $S = R^W = \mathbb{R}[f_1, \ldots, f_n]$ with $\deg f_i = d_i$. Then $E^W = \mathbb{R}[f_1, \ldots, f_n]\langle df_1, \ldots, df_n \rangle$.

We now turn this into an equality of generating functions. We grade $E$ such that $\deg dx_i = 1$. This implies that $\deg df = \deg f$ (for example, $\deg d(x^n) = \deg nx^{n-1}dx = (n-1)+1 = n$.) We compute

$$\sum_{i,p} \dim(E_i^p)^W t^i u^p$$

in two ways. First, from last time's result,

$$\sum_{i,p} \dim(E_i^p)^W t^i u^p = \frac{\prod_i 1 + t^{d_i} u}{\prod_i 1 - t^{d_i}}.$$

But also, copying the proof of Molien's formula,

$$\sum_{i,p} \dim(E_i^p)^W t^i u^p = \frac{1}{|W|} \sum_{w \in W} \frac{\prod_j 1 + tu\lambda_j(w)}{\prod_j 1 - t\lambda_j(w)}.$$

We will get nicer signs if we replace $u$ by $-u$, so we do this now:

$$\frac{\prod_i 1 - t^{d_i} u}{\prod_i 1 - t^{d_i}} = \frac{1}{|W|} \sum_{w \in W} \frac{\prod_j 1 - tu\lambda_j(w)}{\prod_j 1 - t\lambda_j(w)}.$$

We want to find a limit under which this formula behaves particularly nicely. It makes sense to send $u \to 1$, as this makes all the factors with $\lambda_j \neq 1$ cancel out. But then we have better send $t \to 1$ as well, or everything will cancel. If we were just following our noses, we should put $t = 1 + x$ and $u = 1 + rx$ and send $x \to 0$. Having done the computation in advance, it comes out prettier if we put $u = t^{q-1}$ and send $t \to 1$. We get

$$\frac{\prod_i (d_i + q - 1)}{\prod d_i} = \frac{1}{|W|} \sum_{w \in W} \prod_j \begin{Bmatrix} q & \lambda_j(w) = 1 \\ 1 & \lambda_j(w) \neq 1 \end{Bmatrix}.$$

Using that $|W| = \prod d_i$, this simplifies to

$$\prod_i (q + d_i - 1) = \sum_{w \in W} q^{\dim V^w}.$$

**November 27 – Divided Difference Operators.** As usual, let $W$ be a Coxeter group that acts on a vector space $V$. Let $R = \mathrm{Sym}^\bullet V$ and let $\beta \in \Phi^+$ with the associated reflection $t$. Define $\partial_\beta : R \to R$ as follows:

$$\partial_\beta \beta(f) = \frac{f - t \cdot f}{\beta}$$

We verify that $\beta$ divides $f - t \cdot f$: Since $t \cdot (f - t \cdot f) = t \cdot f - f = -(f - t \cdot f)$ and $t \cdot \beta = -\beta$, it follows that $\beta$ divides $f - t \cdot f$, so $\partial_\beta \beta$ is well-defined.

We'll abbreviate $\partial_\beta \alpha_i$ to $\partial_i$ when the meaning is clear.

**Fact 1** $\partial_t$ is $R^t$-linear and, thus, all $\partial_t$ are $S$-linear (where $S = R^W$).

*Proof.* Let $f \in R^t, g \in R$. Then

$$\partial_t(fg) = \frac{fg - t \cdot (fg)}{\beta} = \frac{fg - t(f)t(g)}{\beta} = \frac{fg - f \cdot t(g)}{\beta} = f \partial_\beta g.$$

□

**Fact 2** $\ker \partial_\beta = R^t$

*Proof.* Obvious. □

**Fact 3** $\mathrm{Im} \partial_\beta = R^t$

*Proof.* First, we show $\mathrm{Im} \partial_\beta \supseteq R^t$:

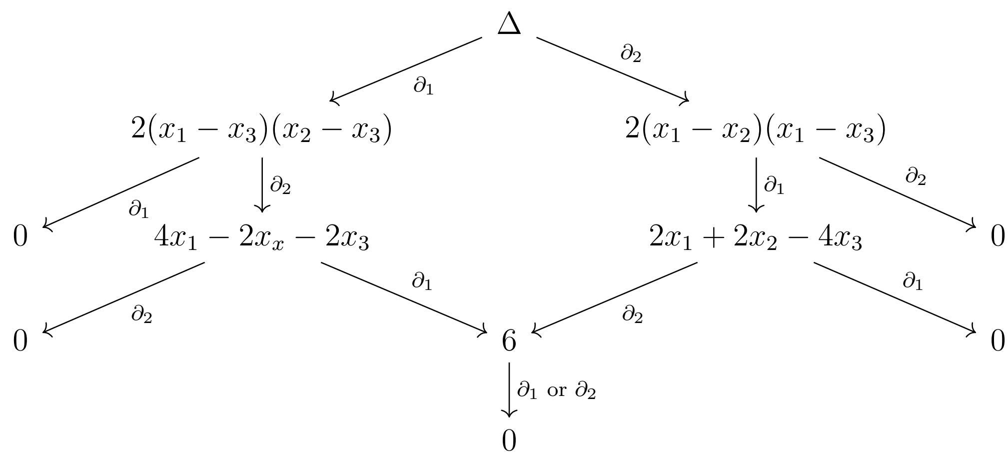
$$t \cdot \partial_t f = t \frac{f - t(f)}{\beta} = \frac{t(f) - f}{-\beta} = \frac{f - t(f)}{\beta} = \partial_\beta f.$$

Now, we show $\mathrm{Im} \partial_\beta \supseteq R^t$. Since $\mathrm{Im} \partial_\beta$ is a $R^t$-submodule, it is enough to check that $1 \in \mathrm{Im} \partial_\beta$:

$$\partial_\beta \left( \frac{1}{2} \beta \right) = \frac{\frac{1}{2}\beta - (-\frac{1}{2}\beta)}{\beta} = 1.$$

□

**Example:** Let $W = A_2$ and $\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. By acting on $\Delta$ repeatedly by $\partial_1$ and $\partial_2$ we get the following polynomials:




In the next few lectures, we will prove:

**Theorem.** Let $W$ be finite (we need this in order for $\Delta$ to be defined).

- The set of non-zero values of $\partial_{i_1} \cdots \partial_{i_t} \Delta$ has size $|W|$ and $R$ is a free $S$-module with this basis.
- More precisely, if a word $s_{i_1} s_{i_2} \cdots s_{i_t}$ is not reduced then $\partial_{i_1} \cdots \partial_{i_t} = 0$.
- If words $s_{i_1} s_{i_2} \cdots s_{i_t}$ and $s_{j_1} s_{j_2} \cdots s_{j_t}$ are reduced with the same product, then $\partial_{i_1} \cdots \partial_{i_t} = \partial_{j_1} \cdots \partial_{j_t} =: \partial_w$ and $\partial_w \Delta \neq 0$
- So our basis of $R$ as an $S$-module is $\{\partial_w \Delta\}$

This theorem will be proved over the next two lectures, but first we will look at its implications. Consider the following identity.

$$\sum_i \dim R_i t^i = \left( \sum_{w \in W} t^{\deg \partial_w \Delta} \right) \left( \sum_j \dim S_j t^j \right)$$

To simplify the identity above, we make the following observation:

$$\sum_{w \in W} t^{\deg \partial_w \Delta} = \sum_{w \in W} t^{\deg \Delta - \ell(w)} = \sum_{w \in W} t^{\ell(w_0) - \ell(w)} = \sum_{u \in W} t^{\ell(u)} \text{ where } u = w^{-1} w_0$$

Using this simplication, we can rewrite the original identity as follows:

$$\frac{1}{(1-t)^n} = \sum_{u \in W} t^{\ell(u)} \cdot \prod \frac{1}{1 - t^{d_i}}$$

By rearranging these terms we getting the following generating function:

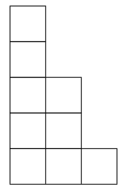
$$\sum_{u \in W} t^{\ell(u)} = \frac{\prod_{i=1}^n (1 - t^{d_i})}{(1-t)^n}$$

Recall that if $(W, \Phi)$ is crystallographic then we have

$$\sum_{w \in W} t^{\ell(w)} = \frac{\prod_{j \geq 2} (1 - t^j)^{h_{j-1} - h_j}}{(1-t)^n} \text{ where } h_j = \#\{\beta \in \Phi^+ : \operatorname{ht}(\beta) = j\}.$$

We deduce $h_j - h_{j-1} = \#\{i : d_i = j\}$. There is a nice way to visualize this: We already know that $\sum h_j = |T| = \sum d_i - 1$. This equality says that $h_j$ and $d_i - 1$ are conjugate partitions. For example, in $B_3$, the number of elements of the root poset at each height is

$(3,2,2,1,1)$ and the number degrees of the invariants are $(6,4,2)$ (the elementary symmetric functions in the $x_i^2$).



We note that the sizes of the rows are $(1,1,2,2,3)$ and the sizes of the columns are $(6-1,4-1,2-1)$.

We make some other remarks:

**Using all reflections doesn't give a basis:** In $S_3$, we have

$$\partial_{e_2-e_1}\Delta - \partial_{e_3-e_1}\Delta + \partial_{e_2-e_1}\Delta = \tfrac{1}{3}(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)\cdot \partial_{e_2-e_1}\partial_{e_3-e_2}\partial_{e_2-e_1}\Delta.$$

**Other bases:** From Nakayama's Lemma, we have a correspondence between $S$-bases for $R$ and $\mathbb{R}$-bases for $R/RS_+$. Since $\partial_\beta$ are $S$-linear maps, they descend to linear maps $R/RS_+ \to R/RS_+$. So, if we choose any homogeneous $F \in R$ with $F \equiv c\Delta \mod RS_+$ for $c \in \mathbb{R}_{\neq 0}$, then $\partial_w F = c\partial_w\Delta$ and $\partial_w F$ will be another $S$-basis for $R$.

**Schubert polynomials** In type $A$, $x_1^{n-1}...x_{n-2}^2x_{n-1} \equiv \frac{1}{n!}\Delta \mod RS_+$ so $\partial_w x_1^{n-1}...x_{n-2}^2x_{n-1} =: \mathfrak{S}_{w^{-1}w_0}$ is another $S$-basis for $R$. These are called Schubert polynomials.

**Flag groups** Finite crystallographic Coxeter groups give us compact Lie groups $K$, and reductive groups $B$. Let $T$ be the maximal torus of $K$, and $B$ the Borel of $G$. The ***flag manifold*** $\mathcal{F}\ell$ is $K/T \cong G/B$.

If we consider the cohomology, we get: $H^\bullet(k/T,\mathbb{R}) \cong H^\bullet(G/B,\mathbb{R}) \cong R/RS_+$. This correspondence doubles degrees: $H^{2i}$ corresponds to $(R/RS_+)_i$ and $H^{2i+1}$ is 0.

In type $A$, the Schubert variety of $X^w$ is $\partial_w x_1^{n-1}...x_{n-2}^2x_{n-1}$. In any type, we have $[X^{s_iw}] = \partial_i[X^w]$. So $[X^w] = \partial_w[X^{id}] = \partial_w[\text{point}]$.

**November 29 – $R$ is free over $S$, part I.** We have our usual notation: $W$ is a finite Coxeter group, it acts on $V$ and $R = \mathrm{Sym}^\bullet(V)$. We put $S = R^W$ and write $R_+$ and $S_+$ for the maximal graded ideals $\bigoplus_{i\geq 1} R_i$ and $\bigoplus_{i\geq 1} S_i$.

We have $S$-module maps $\partial_i : R \to R$. Our long term goal is to show $\{\partial_{i_1}\ldots\partial_{i_t}\Delta \text{ nonzero}\}$ is a free basis for $R$ as $S-$module. Today's goal is to show that $\{\partial_{i_1}\ldots\partial_{i_t}\Delta \text{ nonzero}\}$ spans $R$ as $S$-module.

Let $U$ be a finite dimensional $W$ representation over $\mathbb{R}$, let $\epsilon$ be the operator $\frac{1}{|W|}\sum(-1)^{l(w)}w$ in $\mathbb{R}[W]$, so $\epsilon : U \to U$ is an idempotent projecting onto the anti-symmetric vectors. So we have the decomposition $U = \mathrm{Ker}\,\epsilon \oplus \mathrm{Im}\,\epsilon$.

**Lemma.** In the above notation, $\mathrm{Ker}\,\epsilon = \mathrm{Span}(U^{s_i}, i = 0,\ldots,n)$.

*Proof.* Fix a positive definite symmetric $W$ invariant dot product on $U$, we'll show $\mathrm{Span}(U^{s_i}, i = 0,\ldots,n)$ and $\mathrm{Im}\,\epsilon$ are orthogonal complements. Since $\epsilon$ is self adjoint, its image and kernel are orthogonal complements. We also have the orthogonal decomposition $U = \{1\text{-eigenspace of } s_i\} \oplus \{-1\text{-eigenspace of } s_i\}$. We have $\{1\text{-eigenspace of } s_i\} = U^{s_i}$ and we define $V_i := \{(-1)\text{-eigenspace of } s_i\} = V_i$. Then

$$\mathrm{Span}(U^{s_i})^{\perp} = \bigcap (U^{s_i})^{\perp} = \bigcap V_i = \{v : s_i v = -v \ \forall i\} = \{\text{anti-symmetric vectors}\}. \qquad \square$$

Applying this lemma degree by degree to $R$, we have

$$R = S\Delta \oplus \mathrm{Span}(R^{s_i})_{1\leq i\leq n} = S\Delta \oplus \mathrm{Span}(\partial_i R)_{1\leq i\leq n}.$$

But then

$$\partial_i R = \partial_i(S\Delta \oplus \mathrm{Span}(\partial_j R)_{1\leq j\leq n}) = \mathrm{Span}(S\partial_i\Delta,\ \partial_i\partial_j R)_{1\leq j\leq n}.$$

So we have

$$R = \mathrm{Span}_S(\Delta, \partial_i\Delta, \partial_i\partial_j R)_{i,j=1,\ldots,n}.$$

Continuing in this manner:

$$\mathrm{Span}_S(\partial_{i_1}\ldots\partial_{i_k}\Delta, \partial_{j1}\ldots\partial_{j_N} R)$$

where $k$ runs over $\{0,1,\ldots,N-1\}$ and the indices $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_N$ range from 1 to $n-1$.

**Lemma.** There exits an $N \geq 0$, such that all $\partial_{j_1}\ldots\partial_{j_N}$ act by 0.

*Proof.* Let $g_1,\ldots,g_m$ be the homogeneous generators, take $N > \max\{\deg g_i\}$. For any $h \in R$, we have $h = \sum f_i g_i$ with $f_i \in S$. Then $\partial_{j_1}\ldots\partial_{j_N}\sum f_i g_i = \sum f_i \partial_{j_1}\ldots\partial_{j_N} g_i$. The $\partial_{j_1}\ldots\partial_{j_N} g_i$ have degree $< 0$, so they are zero. $\square$

Wee see that $R = Span_S\{\partial_{j_1}\ldots\partial_{j_k}\Delta\}$ for $k < N$, where $N$ is as in the lemma. We have now proved today's main claim and we pursue corollaries:

**Corollary.** If $N > l(w_0)$, then $\partial_{j_1}\ldots\partial_{j_N} = 0$.

*Proof.* Notice that $\deg(\partial_{j_1}\ldots\partial_{j_k}\Delta) = \deg(\Delta) - k = \ell(w_0) - k = |T| - k$. So we have shown that $R$ is spanned by generators in degrees $\leq \ell(w_0)$ and the proof of the previous lemma shows that $\partial_{j_1}\ldots\partial_{j_N} = 0$ for $N > \ell(w_0)$. $\square$

Note that this is what we would expect from the theorem we are trying to prove, which states that $\partial_{j_1}\ldots\partial_{j_N} = 0$ whenever $s_{j_1}s_{j_2}\cdots s_{j_N}$ is not reduced. We now consider what happens when $N = \ell(w_0)$.

We first show:

**Lemma.**

$$\partial_{i_1}\ldots\partial_{i_{l(w_0)}}(f) = \frac{\sum(-1)^{l(w)}wf}{\Delta}c(i_1,\ldots,i_{l(w_0)})$$

for some $c(i_1,\ldots,i_{l(w_0)}) \in \mathbb{R}$.

*Proof.* Let $f = \sum_{j_1,\ldots,j_k} g_{j_1\ldots j_k}\partial_{j_1}\ldots\partial_{j_k}\Delta$, where $g_{j_1\ldots j_k} \in S$. Then

$$\partial_{i_1}\ldots\partial_{i_{l(w_0)}} f = \sum g_{j_1\ldots j_k}\partial_{i_1}\ldots\partial_{i_{l(w_0)}}\partial_{j_1}\ldots\partial_{j_k}\Delta$$

For $k > 0$, the summand has negative degree and thus vanishes. When $k = 0$, then $\partial_{i_1}\ldots\partial_{i_{l(w_0)}}\Delta$ is degree zero, so it is some scalar. So $\partial_{i_1}\ldots\partial_{i_{l(w_0)}} f$ is a scalar multiple of $g_\emptyset$.

We now look at the right hand side. For $k \geq 1$, we have $\partial_{i_1}\ldots\partial_{i_k}\Delta \in \partial_{i_1}R = R^{s_{i_1}}$. So $\sum(-1)^{l(w)}w$ kills $\partial_{i_1}\ldots\partial_{i_k}\Delta$ for every $k \geq 1$. Thus $\frac{\sum(-1)^{l(w)}wf}{\Delta} = \frac{g_\phi\sum(-1)^{l(w)}w\Delta}{\Delta} = |W|g_\emptyset$.

We have shown that both sides are scalar multiples of $g_\emptyset$, and the scalar on the left hand side is nonzero, so there is some scalar $c$ as claimed. $\square$

We now describe the scalar $c$.

**Lemma.** In the above notation,

$$c(i_1,\dots,i_{l(w_0)}) = \begin{cases} 1 & s_{i_1}\cdots_{i_{l(w_0)}} = w_0 \\ 0 & \text{otherwise} \end{cases}.$$

We abbreviate $c(i_1,\dots,i_{l(w_0)})$ to $c$ from here on.

Let $K = \mathrm{Frac}(R)$ and $L = \mathrm{Frac}(S)$. So $L/K$ is a Galois extension with Galois group $W$.

Let $A = K < W >$ be the skew group algebra. As a set, its elements are of the form $\sum_{w\in W} f_w w$, where $f_w \in K$. The multiplication is given by $(fu)(gv) = f(u \centerdot g)(uv)$, for $f$ and $g \in K$ and $u$ and $v \in W$. The ring $A$ acts on $K$ where $f \in K$ acts by left multiplication and $u \in W$ acts by its action on $L$. So the action of $\partial_\beta$ is by $\beta^{-1}(1-t)$ where $t$ is the reflection in $\beta$. The previous lemma showed that $\prod \alpha_{i_k}^{-1}(1-s_{i_k})$ and $c\Delta^{-1}\sum(-1)^{l(w)}w$ have the same action on $K$.

**Lemma.** $A$ injects in $\mathrm{Hom}_L(K,K)$.

*Proof.* This is a standard Galois theory lemma. See, for example, Corollary 5.15 in Milne's *Fields and Galois Theory* and take $E$ and $F$ (in Milne's notation) to be $L$. □

So we must have $\prod \alpha_{i_k}^{-1}(1-s_{i_k}) = c\Delta^{-1}\sum(-1)^{l(w)}w$ in $A$. Look at the coefficients of $w_0$ on both sides. On the RHS, it is $\frac{c(-1)^{l(w_0)}}{\Delta}$.

For the LHS, the only thing which could give $w_0$ is $s_{i_1}\cdots s_{i_{l(w_0)}}$, so if $s_{i_1}\cdots s_{i_{l(w_0)}} \neq w_0$, the coefficient is 0 and we have $c = 0$ as promised.

Suppose now that $s_{i_1}\cdots s_{i_{l(w_0)}} = w_0$. We need to compute

$$\alpha_{i_1}^{-1}(-s_{i_1})\alpha_{i_2}^{-1}(-s_{i_2})\dots\alpha_{i_{l(w_0)}}^{-1}(-s_{i_{l(w_0)}}).$$

We pull out the factor of $(-1)^{\ell(w_0)}$ and then start moving the group elements to the right by the relation $u\frac{1}{\alpha} = \frac{1}{u(\alpha)}u$.

$$\begin{aligned}
&\frac{1}{\alpha_{i_1}} s_{i_1} \frac{1}{\alpha_{i_2}} s_{i_2} \frac{1}{\alpha_{i_3}} s_{i_3} \frac{1}{\alpha_{i_4}} s_{i_4} \cdots \frac{1}{\alpha_{i_{l(w_0)}}} s_{i_{l(w_0)}} &=\\
&\frac{1}{\alpha_{i_1}} \frac{1}{s_{i_1}\alpha_{i_2}} s_{i_1}s_{i_2} \frac{1}{\alpha_{i_3}} s_{i_3} \frac{1}{\alpha_{i_4}} s_{i_4} \cdots \frac{1}{\alpha_{i_{l(w_0)}}} s_{i_{l(w_0)}} &=\\
&\frac{1}{\alpha_{i_1}} \frac{1}{s_{i_1}\alpha_{i_2}} \frac{1}{s_{i_1}s_{i_2}\alpha_{i_3}} s_{i_1}s_{i_2}s_{i_3} \frac{1}{\alpha_{i_4}} s_{i_4} \cdots \frac{1}{\alpha_{i_{l(w_0)}}} s_{i_{l(w_0)}} &=\\
&\cdots &=\\
&\frac{1}{\alpha_{i_1}} \frac{1}{s_{i_1}\alpha_{i_2}} \frac{1}{s_{i_1}s_{i_2}\alpha_{i_3}} \frac{1}{s_{i_1}s_{i_2}s_{i_3}\alpha_{i_4}} \cdots \frac{1}{s_{i_1}s_{i_2}\cdots s_{i_{\ell(w_0)-1}}\alpha_{i_{\ell(w_0)}}} s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_{\ell(w_0)}}
\end{aligned}$$

By hypothesis, $s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_{\ell(w_0)}} = w_0$. Since the word $s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_{\ell(w_0)}}$ is reduced, the product on the left is $\prod_{\beta\in\mathrm{inv}(w_0)}\beta^{-1} = \Delta^{-1}$. So (putting back the power of $(-1)$) we have

$$c(-1)^{l(w_0)}\Delta^{-1} = (-1)^{\ell(w_0)}\Delta^{-1}$$

and $c = 1$ as claimed. □

**December 1 – $R$ is free over $S$, part II.** We have again the usual $W$ (finite), $V$, $R$, $R_+$, $S$, $S_+$, and $\Delta$. Last time, we showed $\{\partial_{i_1}\dots\partial_{i_t}\Delta\}$ span $R$ as an $S$ module. We also showed if $t > \ell(w_0)$, $\partial_{i_1}\dots\partial_{i_t} = 0$, and if $t = \ell(w_0)$,

$$\partial_{i_1}\dots\partial_{i_t} = \begin{cases} 0 & s_{i_1}\dots s_{i_t} \neq w_0 \\ \frac{\Sigma(-1)^{\ell(w_0)}w}{\Delta} & s_{i_1}\dots s_{i_t} = w_0 \end{cases}.$$

Also $\partial_i^2 = 0$.

These foreshadow the general claim: $\partial_{i_1} \ldots \partial_{i_t} = 0$ if $s_{i_1} \ldots s_{i_t}$ is not reduced, and if $s_{i_1} \ldots s_{i_t}$ and $s_{j_1} \ldots s_{j_t}$ are reduced words for $w$, then $\partial_{i_1} \ldots \partial_{i_t} = \partial_{j_1} \ldots \partial_{j_t}$, which we will then call $\partial_w$.

We will leverage November 29th's results and a previous homework problem to prove this claim. We start by recalling proving the braid move lemma from the homework.

**Lemma.** If $w = s_{i_1} \ldots s_{i_t} = s_{j_1} \ldots s_{j_t}$, both reduced, then we can turn one into other by a sequence of braid moves.

*Proof.* Induct on $t$. The base case $t = 0$ gives us the empty word. Now, if $i_1 = j_1$, we can change $s_{i_2} \ldots s_{i_t}$ to $s_{j_2} \ldots s_{j_t}$ by induction. Otherwise, we put $i = i_1$, $j = j_1$, and $m_{ij} = m$. Looking at $s_i \ldots s_{i_t}$, we have $s_i \in \mathrm{Inv(w)}$, and likewise, $s_j \in \mathrm{Inv(w)}$. So $wD$ is on the other side of $\alpha_i$ and $\alpha_j$ from $D$. So it is also on the other side of $\beta^\perp$ for all $\beta \in \Phi_{\{i,j\}}$. In other words, $s_i$, $s_i s_j s_i$, $s_i s_j s_i s_j s_i$, etc. are all in Inv(w). So then, $s_i \in \mathrm{Inv(w)}$, $s_j \in \mathrm{Inv(s_i^{-1}w)}$, and $s_i \in \mathrm{Inv((s_is_j)^{-1}w)}$. Continuing this, we can put $w = \underbrace{s_i s_j s_i \ldots}_{m} u$, where $\ell(w) = m + \ell(u)$. Choosing a reduced word for $u$, $s_{k_1} \ldots s_{k_n}$, we can then get from $s_i \ldots s_{i_t}$ to $(s_i s_j s_i \ldots) s_{k_1} \ldots s_{k_n}$ via our induction, which can be turned into $(s_j s_i s_j \ldots) s_{k_1} \ldots s_{k_n}$ with a braid move, which finally can be changed to $s_j \ldots s_{j_t}$, again by our induction. □

Now, we move on to proving today's main claim. Let $I \subseteq [n]$, with $(w_0)_I$ the longest element in the parabolic subgroup $W_I$. Let $s_{i_1} \ldots s_{i_t}$ and $s_{j_1} \ldots s_{j_t}$ be reduced words for $(w_0)_I$.

**Proposition.** $\partial_{i_1} \ldots \partial_{i_t} = \partial_{j_1} \ldots \partial_{j_t}$.

*Proof.* We have $V_I \oplus V_I^\perp$, where $V_I := \mathrm{Span}(\alpha_i | i \in I)$. Our group $W$ acts trivially on $V_I^\perp$, and acts the usual way on $V_I$. Choosing a basis $z_1, \ldots, z_k$ for $V_I^\perp$, we get $R \cong R_I[z_1, \ldots, z_k]$, and for $\beta$ in $\Phi_I$, $\partial_\beta$ acts on $R$ through action on $R_I$. So our claim follows from having the claim for $W_I$ acting on $R_I$. □

In particular, we already know this for $\underbrace{\partial_i \partial_j \partial_i \ldots}_{m_{ij} \text{ times}} = \underbrace{\partial_j \partial_i \partial_j \ldots}_{m_{ij} \text{ times}}$. From our earlier lemma, if we have $s_{i_1} \ldots s_{i_t}$ and $s_{j_1} \ldots s_{j_t}$ both reduced with the same product, we can turn one into the other with braid moves. We have now proved that, if $s_{i_1} s_{i_2} \cdots s_{i_t}$ and $s_{j_1} s_{j_2} \cdots s_{j_t}$ are reduced words with the same product, then $\partial_{i_1} \ldots \partial_{i_t} = \partial_{j_1} \ldots \partial_{j_t}$.

Now, take $s_{i_1} \ldots s_{i_t}$ not reduced. Put $t_k = s_{i_1} \ldots s_{i_k} \ldots s_{i_1}$. By our hypothesis, some $t_k = t_\ell$ for $k < \ell$. Take $k$ to be maximal. We will show $\partial_{i_k} \ldots \partial_{i_t} = 0$. We will reindex to $k = 1$, so $s_{i_1} \ldots s_{i_m}$ is not reduced but $s_{i_2} \ldots s_{i_m}$ is reduced. So $s_{i_1}$ is an inversion of $s_{i_2} \ldots s_{i_m}$. This gives us a reduced word $s_{j_2} \ldots s_{j_m}$ for $s_{i_2} \ldots s_{i_m}$ with $j_2 = i_1$. So,

$$\partial_{i_1}(\partial_{i_2} \ldots \partial_{i_m}) = \partial_{i_1}(\partial_{j_2} \ldots \partial_{j_m}) = \partial_{i_1}^2(\text{something}) = 0.$$

We now know that $\{\partial_{i_1} \cdots \partial_{i_t} \Delta \text{ nonzero}\}$ is $\{\partial_w \Delta\}$, which has size $|W|$.

So $R$ has an $S-$spanning of size $|W|$. As we pointed out at the end of November 3, this must be a free $S$-basis for $R$.

**December 4 – Rings of invariants are polynomial, second proof.** We will now prove for a second time that $S$ is a polynomial ring. This will follow immediately from

**Theorem** (Bourbaki, Lie Groups and Lie Algebras V §5)**.** Let $R = k[x_1, \cdots, x_n]$ of characteristic zero[3]. Let $S$ be a graded sub-$k$-algebra of $R$ such that $R$ is free of finite rank as an $S$-module. Then $S \cong k[y_1, \cdots, y_n]$.

We put $R_+ = \oplus_{i\geq 1} R_i$ and $S_+ = \oplus_{j\geq 1} S_j = S \cap R_+$.

The key property of free modules that we will use is this: If $M$ is a free $S$-module and $f \in M$ with $f \notin S_+M$, then $\exists \lambda : M \to S$ a map of $S$-modules such that $\lambda(f) \notin S_+$. Proof: Let $M = Sg_1 \oplus \cdots \oplus Sg_n$ and write $f = \sum c_i g_i$ with $c_i \in S$. Then some $c_j \notin S_+$. Let $\lambda$ take coefficient of $g_j$. If $M$ is graded and $f$ homogeneous, we can take $\lambda$ graded. So $\lambda(f) \in S_0 = k$ and is not 0.

Consider $RS_+ \otimes_R R/R_+ = RS_+/R_+S_+$. This is a graded $R/R_+ = k$ vector space. Let $\bar{f}_1, \cdots, \bar{f}_p$ be a homogeneous basis; lift to $f_1, \cdots, f_p$ homogeneous elements of $S$. So as an $R$-module, $RS_+ = \langle f_1, \cdots, f_p\rangle$ by Nakayama.

**Claim.** $f_1, \cdots, f_p$ generate S.

In the case of rings of invariants, we already proved this on November 3 (our second proof of finite generation). The proof below just repeats this argument, replacing the averaging operator with an abstract map $\lambda_0$.

*Proof.* Let $\lambda_0 : R \to S$ be an $S$-module map with $\lambda_0(1) = 1$. Let $P$ be the sub-$k$-algebra of $S$ generated by $f_1, \ldots, f_p$. We show by induction on $j$ that $P_j = S_j$. Let $j > 0$ and $g \in S_j$. So $g = \sum h_j f_j$ for some $h_j \in R$. We have $g = \lambda_0(g) = \sum f_j \lambda_0(h_j)$. So $g$ is in the ring generated by $f_j$ (in $P$ by definition) and $\lambda_0(h_j)$ (in $P$ by induction). □

So we know that $S$ is generated by the $f_i$, and we want to show that they are algebraically independent. Suppose for contradiction that there exists $H \in k[y_1, \cdots, y_p]$, $H \neq 0$, with $H(f_1, \cdots, f_p) = 0$. Set $h_i = \frac{\partial H}{\partial y_i}|_{y_j = f_j} \in S \subset k[x_1, \cdots, x_n]$. Note at least one $h_i \neq 0$.

We have $H(f_1, \cdots, f_p) = 0$ so $\begin{bmatrix} h_1 & \cdots & h_p \end{bmatrix} \left[\frac{\partial f_i}{\partial x_j}\right]_{1\leq i\leq p, 1\leq j\leq n} = 0$. We introduce the shorthands

$$D = \left[\frac{\partial f_i}{\partial x_j}\right] \qquad \vec{h} = [h_i]$$

so this relation is

$$\vec{h}^T D = 0.$$

We also need a basic observation, due to Euler:

**Proposition.** If $f \in k[x_1, \cdots, x_n]$ is homogeneous then $\sum x_j \frac{\partial f}{\partial x_j} = (\deg f) f$.

*Proof.* $\sum_j x_j \left(\frac{\partial x_1^{d_1} \cdots x_n^{d_n}}{\partial x_j}\right) = \sum_j d_j (x_1^{d_1} \cdots x_n^{d_n}) = (\sum d_j) x_1^{d_1} \cdots x_n^{d_n}$. □

$\vec{h}^T D = \vec{0}$ Putting $\vec{x} = [x_i]$ and $\vec{y} = [(\deg f_j) f_j]$, we can also encode this as a matrix equation:

$$D\vec{x} = \vec{y}.$$

We first sketch the proof.

[3]Actually, the result is true without this hypothesis; see the end of this day's notes.

*Sketch of proof.* **Case I:** If all $D_{ij} \in RS_+$ then all entries of $D\vec{x}$ lie in $R_+S_+$. So $f_i \in R_+S_+$, and $\bar{f}_i = 0$. But $\bar{f}_i$ is a basis for $RS_+/R_+S_+$.

**Case II:** If $D_{ij} \notin RS_+$, then $\exists \lambda : R \to S$ with $\lambda(D_{ij}) \notin S_+$. Then $\vec{h}^T \lambda(D) = 0$ gives linear relations between the $h_i$. There are so many linear relations that we contradict our knowledge that some of the $h_i$ are nonzero □

We now give the details:

*Proof.* Set $K = \langle h_1, \cdots, h_p \rangle \subset S$. Note $K \neq (0)$. Let $\bar{h}_i$ be the image of $h_i$ in $K \otimes_S S/S_+$. Let $\bar{h}_1, \cdots, \bar{h}_m$ be a basis of $K \otimes_S S/S_+$ By Nakayama, $K = \langle h_1, \cdots, h_m \rangle$ as an $S$-module. (In particular, $m > 0$.) For $j > m$, let $h_j = \sum_{i=1}^{m} G_{ij} h_i$ with $G_{ij} \in S$. So we have $\vec{h} = \begin{bmatrix} Id_m \\ G \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots h_m \end{bmatrix} = \begin{bmatrix} Id_m \\ G \end{bmatrix} \vec{h}_{gen}$.

So $\vec{h}_{gen}^T \begin{bmatrix} Id_m & G^T \end{bmatrix} D = \vec{h}^T D = \vec{0}$. Put $\tilde{D} = \begin{bmatrix} Id_m & G^T \end{bmatrix} D$, so $\vec{h}_{gen}^T \tilde{D} = 0$.

Case I: All entries of $\tilde{D}$ are in $RS_+$. Then all entries of $\tilde{D}\vec{x}$ are in $R_+S_+$. We have $\tilde{D}\vec{x} = \begin{bmatrix} Id_m & G^T \end{bmatrix} \vec{y}$. So we deduce that $(\deg f_1) f_1 + \sum G_{j1} (\deg f_j) f_j \in R_+S_+$. But this shows that $(\deg f_1)\bar{f}_1 + \sum \bar{G_{j1}} (\deg f_j) \bar{f}_j = 0$ in $RS_+/R_+S_+$. This gives a nontrivial linear relation between the $\bar{f}_i$, a contradiction.

Case II: Some $\tilde{D}_{ij} \notin RS_+$. Then $\exists \lambda : R \to S$ with $\lambda(\tilde{D}_{ij}) \notin S_+$. Since the entries of $\vec{h}_{gen}$ are in $S$, we have $\lambda(\vec{h}_{gen}^T \tilde{D}) = \vec{h}_{gen}^T \lambda(\tilde{D})$ so $\vec{h}_{gen}^T \lambda(\tilde{D}) = 0$ and we get that there is an $S$-linear relation between $h_1, \ldots, h_m$, not all of whose coefficients are in $S_+$. But then mapping to $K/S_+K$ gives a nontrivial linear relations between $\bar{h}_1, \ldots, \bar{h}_m$, contradicting that the $\bar{h}_i$ are a basis of $K/S_+K$. □

We remark on an alternate proof, due to Neil Strickland.[4] Take the Koszul complex, this is a free resolution $0 \to F_n \to \cdots F_1 \to F_0 \to k$ of $k$ over $R$. Since $R$ is assumed a free $S$-module, the $F_i$ are also free $S$-modules and this is a finite length free resolution of $k$ as an $S$-module. But by a Theorem of Serre (see, for example, Matsamura's *Commutative Ring Theory*, Theorem 19.2), if $S$ is a graded local ring over which the residue field has a finite free resolution, then $S$ is regular. As we discussed before, a graded regular ring is a polynomial ring.

**December 6 – Coxeter Elements.** Let $W$ be a Coxeter group, $s_1, \ldots, s_n$ are simple generators. A Coxeter element of $W$ is an element of the form $s_1 \cdots s_n$, for some ordering of simple generators.

**Example.** In $A_3 \cong S_3$, we have

$$(12)(23) = (123), (23)(12) = (132).$$

In some sources, this is called a standard Coxeter element, and a Coxeter element is defined as anything conjugate to a standard Coxeter element.

[4]https://mathoverflow.net/questions/282780

Here we note that $s_1 \cdots s_n$ is reduced. And we have

$$\begin{aligned} \beta_1 &= \alpha_1 \\ \beta_2 &= s_1\alpha_1 \equiv \alpha_2 \mod \mathbb{R}\alpha_1 \\ &\vdots \\ \beta_n &= s_1 \cdots s_{n-1}\alpha_n \equiv \alpha_n \mod \mathbb{R}\{\alpha_1, \dots, \alpha_{n-1}\} \end{aligned}$$

So $\beta_i$ are linearly independent and in particular distinct. Since the words $s_{i_1} \cdots s_{i_n}$ and $s_{j_1} \cdots s_{j_n}$ for Coxeter elements are reduced, they give the same Coxeter element only if they differ by interchanging commuting generators.

**Example.** In $A_3 = S_4$, we have

$$\begin{aligned} s_1 &= (12) \\ s_2 &= (23) \\ s_3 &= (34). \end{aligned}$$

And the collection of Coxeter elements are

$$s_1s_2s_3,\ s_3s_2s_1,\ s_1s_3s_2 = s_3s_1s_2,\ s_2s_1s_3 = s_2s_3s_1.$$

So there are 4 Coxeter elements in $A_3$.

Now let $\Gamma$ be the Coxeter graph, with vertices $1, \dots, n$, and edges connecting $i \to j$ if $s_is_j \neq s_js_i$. An ordering of $s_1, \dots, s_n$ gives an acyclic orientation of $\Gamma$, where $i \to j$, if $s_i$ is before $s_j$ in reduced words of $c$. Interchanging commuting generators doesn't change this orientation, since there is no edge between commuting generators, so this is a map from

**Proposition.** Coxeter elements correspond bijectively to acyclic orientations of the Coxeter graph.

*Proof.* Surjectivity: Any acyclic orientation extends to a total order.

Injectivity: We induct on $n$. Assume $s_{i_1} \cdots s_{i_n}$ and $s_{j_1} \cdots s_{j_n}$ give the same orientation $\mathcal{O}$. Let $i_1$ and $j_1$ be sources of $\mathcal{O}$. We assume $j_k = i_1$, then $j_1, \dots, j_{k-1}$ commutes with $i_1$. So we get $s_{j_1} \cdots s_{j_{k-1}} s_{j_k} \cdots s_{j_n} = s_{i_1} s_{j_1} \cdots s_{j_{k-1}} s_{j_{k+1}} \cdots s_{j_n}$. ∎

**Corollary.** If $\Gamma$ is a tree with $n$ vertices, then there are $2^{n-1}$ Coxeter elements.

**Proposition.** If $\Gamma$ is a tree (or a forest), then all of Coxeter elements are conjugate.

We note that the conjugation $s_1^{-1}(s_1 \cdots s_n)s_1 = s_2 \cdots s_n s_1$, corresponds to reversing the source at 1 and turning it into a sink. In other words, we claim that we can get from any given $\mathcal{O}_1$ to any other $\mathcal{O}_2$, by taking sink-source and source-sink reversals.

*Proof of the Claim.* We take the induction on $n$. Let $j$ be a leaf of $\Gamma$, and let $k$ be its neighbor. We let $\Gamma'$ be induced subtree on $[n] - j$ (See the picture below). Let $\mathcal{O}_1'$ and $\mathcal{O}_2'$ be the restriction of $\mathcal{O}_1$ and $\mathcal{O}_2$ to $\Gamma'$. Then by induction, we could turn $\mathcal{O}_1'$ into $\mathcal{O}_2'$ by a series of reversals.

What happens when we try to extend this series to $\Gamma$? If we cannot take reverse at $k$, we can insert reversal at $j$ first in order to permit the reverse at $k$. And if $j - k$ edge is wrong at the end of the process, we can reverse at $j$ one more time to make our process keep going. So, inserting these extra reversals at $j$, we can transform $\mathcal{O}_1$ to $\mathcal{O}_2$. ∎

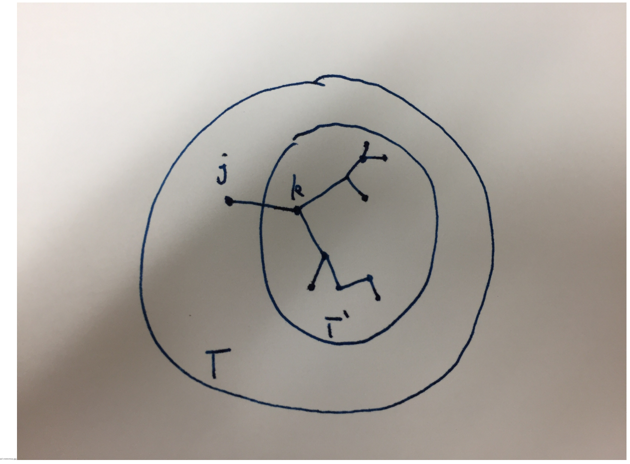



FIGURE 30. $\Gamma$ and $\Gamma'$

We prove a lemma for future use:

**Lemma.** If $\alpha_1, \dots, \alpha_n$ and $\alpha_1^\vee, \alpha_n^\vee$ are bases for $V$ and $V^\vee$, then $c$ acts on $V^\vee$ without eigenvalue 1.

More strongly, if $s_1 \cdots s_k x = x$ for some $x \in V^\vee$. We have

$$\langle x, \alpha_1 \rangle = \cdots = \langle x, \alpha_k \rangle = 0.$$

*Proof.* We still induct on $k$. Assume $s_1 \cdots s_k x = x$, then we get

$$x \equiv s_2 \cdots s_k x = s_1 x \equiv x, \quad \mod \operatorname{Span}(\alpha_1^\vee, \dots, \alpha_{k-1}^\vee).$$

So $s_2 \cdots s_k x \in \operatorname{Span}(\alpha_2^\vee, \dots, \alpha_k^\vee)$, and $s_1 x - x \in \operatorname{Span}(\alpha_1^\vee)$. In this way, since $\alpha_1^\vee, \dots, \alpha_k^\vee$ are linearly independent, we have

$$s_1 x - x = s_2 \cdots s_k x - x = 0.$$

Thus $\langle x, \alpha_1 \rangle$ and $\langle x, \alpha_2 \rangle = \cdots \langle x, \alpha_k \rangle = 0$.

□

Suppose $\Gamma$ is bipartite. We label the vertices $+$ and $-$, such that all edges are between vectices of different signs. We take $t_+ = \prod_{i \text{ is } +} s_i$, and $t_- = \prod_{i \text{ is } -} s_i$. Since all the $+$ generators commute, and likewise for the $-$ generators, these products are well defined. Then $t_+ t_-$ is a Coxeter element, which is called as "the" bipartite Coxeter element. If we order the $+$ rows of the Cartan matrix together, and the $-$ rows together, then the Cartan matrix takes the form

$$A = \begin{bmatrix} 2\mathrm{Id} & -B \\ -B^T & 2\mathrm{Id} \end{bmatrix}.$$

Here the entries of $B$ are positive, and we have chosen to use a symmetric Cartan matrix.

So how do $t_+$ and $t_-$ act on $\alpha_i$ and $\omega_i$, where $\omega_i$ is the dual basis of $\alpha_i^\vee$? If $i$ is $+$, then

$$t_+ \alpha_i = -\alpha_i.$$

And if $j$ is $-$, then

$$\begin{aligned} s_{i_1} \cdots s_{i_k} \alpha_j &= s_{i_1} \cdots s_{i_{k-1}} (\alpha_j - A_{i_k j} \alpha_{i_k}) \\ &= s_{i_1} \cdots s_{i_{k-2}} (\alpha_j - A_{i_{k-1} j}) \alpha_{i_{k-1}} - A_{i_k j} \alpha_{i_k} \\ &\cdots \alpha_j - \sum_r A_{i_r j} \alpha_{i_r}, \end{aligned}$$

So $t_+$ acts on the $\alpha_i$ basis by

$$\begin{bmatrix} -\mathrm{Id} & B \\ 0 & \mathrm{Id} \end{bmatrix}$$

and $t_-$ acts by

$$\begin{bmatrix} \mathrm{Id} & 0 \\ B^T & -\mathrm{Id} \end{bmatrix}.$$

The action on the $\omega$ basis is by

$$\begin{bmatrix} -\mathrm{Id} & 0 \\ B^T & \mathrm{Id} \end{bmatrix} \text{ and } \begin{bmatrix} \mathrm{Id} & B \\ 0 & -\mathrm{Id} \end{bmatrix}.$$

**December 8 – The Coxeter Plane.** Let $W$ be a Coxeter group with generators $s_1, \dots, s_n$, $\Gamma$ be it's Coxeter diagram and let $\Gamma$ be connected and bipartite. Fix a $\{+, -\}$ colouring of $\Gamma$.

Let $t_+ = \prod_{i \text{ is } +} s_i$ and $t_- = \prod_{i \text{ is } -} s_i$ and $\alpha_i$ be a basis of $V$.

Then, $\langle t_+, t_1 \rangle$ is abstractly $I_2(h)$ for $h$ the order of $t_+ t_-$(possibly infinite).

If $\Gamma$ is a tree, all Coxeter elements are conjugate and so $h$ is the order of any Coxeter element, called the Coxeter number.

We wish to find a 2-plane $H \subseteq V^\vee$ on which $\langle t_+, t_- \rangle$ acts be the standard representation of $I_2(n)$, with $t_+$ and $t_-$ its simple generators.

We review some linear algebra we will use here.

**Lemma.** Let $B$ be an $n_+ \times n_-$ real matrix. Then, $\exists$ nonzero vectors $u_+, u_-$ and a scalar $\sigma \geq 0$ such that $Bu_- = \sigma u_+$, $B^\top u_+ = \sigma s_-$.

**Definition.** In the above lemma, $u_+$ and $u_-$ are called the **singular vectors** of $B$ and $\sigma$ is the corresponding **singular value**.

Then, we get that $B^\top B u_- = \sigma^2 u_-$, so $u_-$ is an eigenvector of $B^\top B$ with eigenvalue $\sigma^2$.

**Lemma.** With the setup of the previous lemma, if all entries of $B$ are $\geq 0$, then we can take $u_+, u_-$ to have nonnegative real entries. In addition, let $\Gamma$ be the bipartite graph with edge $(i, j)$ if $B_{ij} > 0$. Then, if all $B_{i,j} \geq 0$ and $\Gamma$ connected, we can take $u_+, u_-$ to have positive real entries.

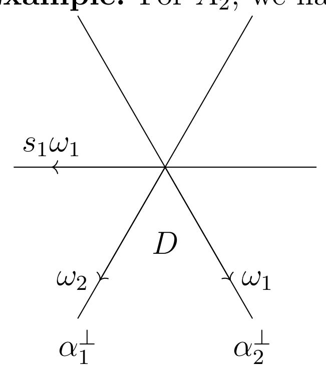
Resuming our previous discussion from before the linear algebra aside, we note that we may take our Cartan matrix to be symmetric and group our $+$ generators before our negative generators. Then, the Cartan matrix is of the form

$$A = \left[\begin{array}{ccc|ccc} 2 & & & & & \\ & \ddots & & & -B & \\ & & 2 & & & \\ \hline & & & 2 & & \\ & -B^\top & & & \ddots & \\ & & & & & 2 \end{array}\right]$$

and the action on the $w_i^\vee$ basis of $t_\pm$ is

$$t_+ = \left[\begin{array}{c|c} 1 & 0 \\ \hline B^\top & -1 \end{array}\right], t_- = \left[\begin{array}{c|c} -1 & B \\ \hline 0 & 1 \end{array}\right]$$

**Example.** For $A_2$, we have $s_1(w_1) = -w_1 + w_2$ and $s_1(w_2) = w_2$

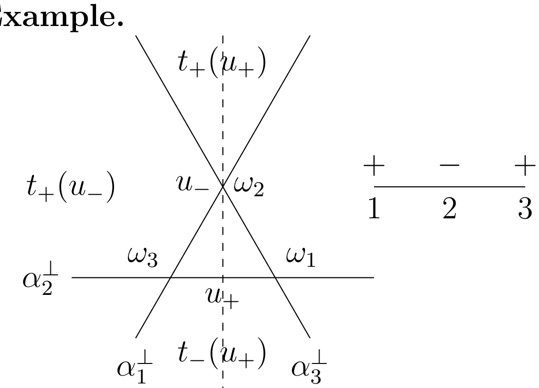



Now, let $n_+$, $n_-$ be the sizes of $+$ and $-$ sets of $\Gamma$ and apply the linear algebra lemma above: $Bu_- = \sigma u_+$, $Bu_+ = \sigma u_{n_-}$, $\sigma > 0$, $u_\pm$ have positive real entries.

We identify $u_+$ with the vector $\left[\begin{smallmatrix} u_+ \\ 0 \end{smallmatrix}\right]$ and $u_-$ with the vector $\left[\begin{smallmatrix} 0 \\ u_- \end{smallmatrix}\right]$ Then, $t_+u_+ = u_+ - \sigma u_-$ and $t_+u_- = u_-$.

So $t_+$ acts on $H := \text{span}(u_+, u_-)$ by $\left[\begin{smallmatrix} -1 & 0 \\ \sigma & 1 \end{smallmatrix}\right]$ and $t_-$ acts by $\left[\begin{smallmatrix} 1 & \sigma \\ 0 & -1 \end{smallmatrix}\right]$.

**Example.**




Here, $t_+$ is acting by rotation (reflection across $\alpha_1^\perp$ and $\alpha_3^\perp$) and $t_-$ by reflection across $\alpha_2^\perp$

$t_+$ and $t_-$ act on $H$ by reflections and $t_+t_- \begin{cases} \text{has order } \infty & \text{if } \sigma \geq 2 \\ \text{acts like rotation by } \theta & \text{if } \sigma = 2\cos\theta \end{cases}$

By the linear algebra lemma, $\sigma > 0$ and $u_+$ and $u_-$ are in the interior of $\text{Span}_{\mathbb{R}_{\geq 0}}(w_i)_{i \text{ is } +}$ and $\text{Span}_{\mathbb{R}_{\geq 0}}(w_i)_{i \text{ is } -}$ respectively.

If $\sigma \geq 2$ then $t_+t_-$ has infinite order and $t_+$, $t_-$ act as simple generators of $I_2(\infty)$ on $H$.

Let $\sigma < 2$, $\sigma = 2\cos\theta$. We claim that $\theta \in \pi\mathbb{Q}$.

*Proof.* If not, some power of $t_+t_-$ will rotate $\mathrm{Span}_{\mathbb{R}\geq 0}(u_+, u_-)$ to overlap itself but not be equal. (Notice $D \cap H = \mathrm{Span}_{\geq 0}(u_+, u_-)$, $D^0 \cap H = \mathbb{R}_{>0}u_+ + \mathbb{R}_{>0}u_-$).

Then, we have $(t_+t_-)^* D^0 \cap H \neq \emptyset$ without $(t_+t_-)^* = \mathrm{Id}$. □

Letting $\theta = \frac{p}{q}\pi$, $(t_+t_-)(D^0 \cap H)$ is disjoint from $D^0$ for $1 \leq p \leq q-1$ and is $D^0 \cap H$ for $p = q$. So $t_+t_-$ has order $q = h$.

The $2h$ cones $(D \cap H)$, $t_+(D \cap H)$, $t_+t_-(D \cap H)$ are all disjoint in $H$.

So $t_+t_-$ moves 2 steps around; $\theta = \frac{\pi}{h}$

**Appendix: Proof of claims about singular vectors**

Let $B$ be an $m \times n$ real matrix. Let $u$ be a vector in $\mathbb{R}^n$ of norm 1 which maximizes $|Bu|$ and put $Bu = \sigma v$ for $v$ a unit vector and $\sigma \geq 0$. I claim that $B^T v = \sigma u$. Proof: Since $|Bu|^2 = u^T B^T B u$ is maximized at $u$, the first order variation of $u^T B^T B u$ must be 0. Let $u'$ be perpendicular to $u$. Then $(u + \epsilon u')^T B^T B(u + \epsilon u') = u^T B^T B u + 2\epsilon u'^T B^T B u + O(\epsilon^2)$. So $u'^T B^T B u = 0$. Since we $v$ is proportional to $Bu$, we have $u'$ perpendicular to $B^T v$. This is true for all $u'$ perpendicular to $u$, so we deduce that $B^T v$ is proportional to $u$, say $B^T v = \tau u$. Then $v^T B u = v^T \sigma v = \sigma$ and $v^T B u = \rho u^T u = \rho$ so $\rho = \sigma$.

Now, suppose that all the entries of $B$ are nonnegative. Let $u = (u_1, \ldots, u_n)$ be any vector in $\mathbb{R}^n$ and let $\bar{u} = (|u_1|, |u_2|, \ldots, |u_n|)$. Then $|u| = |\bar{u}|$ and $|B\bar{u}| \geq |Bu|$, so the largest value of $Bu$ occurs at a vector in $\mathbb{R}^n_{\geq 0}$.

Finally, assume that all $B_{ij} \geq 0$ and make a bipartite graph $\Gamma$ where there is an edge $(i, j)$ if $B_{ij} > 0$. Let $u$ and $v$ be as above. We claim that all $u_i$ and $v_j$ are strictly positive if $\Gamma$ is connected. If not, let $I$ be the set of $i$ for which $u_i > 0$ and let $J$ be the set of $j$ for which $v_j > 0$. Then the induced subgraph of $\Gamma$ on $I \cup J$ is a union of connected components of $\Gamma$. So, when $\Gamma$ is connected, all $u_i$ and all $v_j$ are $> 0$.

**December 11 – Coxeter eigenvalues and invariant theory.** (The first part of today's talk was meant to be in the previous lecture.) Let $W$ be a finite Coxeter group whose Coxeter graph is disconnected and let $H = \mathrm{Span}(u_+, u_-)$ be the Coxeter plane. Let $L_1$, $L_2$, …, $L_h$ be the $h$ lines in $H$ fixed by the $h$ reflections of $\langle t_+, t_- \rangle \cong I_2(h)$. Let $n_+$ and $n_-$ be the number of $+$ and $-$ vertices, so $n = n_+ + n_-$.

**Lemma.** For any $\beta \in \Phi$, the intersection $\beta^\perp \cap H$ is one of the $L_i$.

*Proof.* First of all, we claim that $H$ is not contained in $\beta^\perp$. This is because positive combinations of $u_+$ and $u_-$ lie in $D^\circ$, and $D^\circ$ is disjoint from all $\beta^\perp$. Thus, $H \cap \beta^\perp$ is a line $L$.

We now claim that $L$ is one of the $L_i$. If not, it lies in between two of the $L_i$. Acting by $t_+$ and $t_-$, we may assume it lies between $\mathrm{Span}\, u_+$ and $\mathrm{Span}\, u_-$. But then $L$ meets $D^\circ$, and $\beta^\perp$ cannot meet $D^\circ$, a contradiction. □

We will now consider which $\beta^\perp$'s can contain which lines. Letting $\beta^\perp$ be the fixed plane of $t$, we have $\beta^\perp \cap H = \mathbb{R}u_+$ if and only if $tu_+ = u_+$. From a homework problem, the stabilizer of $u_+$ is $\langle s_i \rangle_{i \text{ is } +}$. Since all the $s_i$ where $i$ is $+$ commute, this group is simply $(\mathbb{Z}/2)^{n_+}$. So the only reflections in this group are $\{s_i\}_{i \text{ is } +}$. We see that there are $n_+$ reflecting hyperplanes for which $\beta^\perp \cap H = \mathbb{R}u_+$. More generally, as we travel through $L_1$, $L_2$, …, $L_h$, the number of reflecting hyperplanes through these lines are $n_+$, $n_-$, $n_+$, $n_-$ .... (If $h$ is odd, this shows $n_+ = n_-$.) We deduce that

$$|\Phi| = |T| = h\frac{n_+ + n_-}{2} = \frac{hn}{2}.$$

As a corollary, we obtain an almost canonical reduced word for $w_0$: We claim that $\overbrace{t_+t_-t_+t_-\cdots}^{h}$ is a reduced word for $w_0$. Proof: This word carries the positive span of $u_+$ and $u_-$ to its negation, so it carries points of $D^\circ$ to $-D^\circ$, so $\overbrace{t_+t_-t_+t_-\cdots}^{h} = w_0$. The length of $w_0$ is $|T|$, and we have just shown this is the number of generators in the word.

**Connection to invariant theory** (We now reach where today's lecture was meant to start.) Let $h$ be the Coxeter number and let the eigenvalues of $h$ on $V \otimes \mathbb{C}$ be $\exp(2\pi i m_j/h)$ for $0 \le m_j < h$. Since we showed that $c$ does not fix any nonzero vector, we actually have $0 < m_j < h$.

Define $R$ and $S$ as usual and let $S = \mathbb{R}[f_1, \ldots, f_n]$ with $\deg f_i = d_i$.

Our final theorem for the course is that (after potentially reordering), we have $m_i = d_i - 1$.

We remark that if $u$, $v$ is an orthonormal basis for some 2-plane $H$, and $c$ orthogonally rotates this 2-plane by $\theta$, then $c$ acts on $H \otimes \mathbb{C}$ with eigenvalues $e^{\pm i\theta}$ and eigenvectors $u \pm iv$.

In particular, let $H$ be the Coxeter plane, and let $(u, v)$ be an orthonormal basis of it. Let $z_1 = u + iv$, so $z_1$ is an $\exp(2\pi i/h)$ eigenvector of $c$.

**Lemma.** For any $\beta \in \Phi$, we have $\langle \beta, z_1 \rangle \neq 0$.

*Proof.* We have $\langle \beta, u + iv \rangle = \langle \beta, u \rangle + i\langle \beta, v \rangle$. Since $\langle \beta, u \rangle$ and $\langle \beta, v \rangle$ are both real, this can only be 0 if $\langle \beta, u \rangle = \langle \beta, v \rangle = 0$. But $u$ and $v$ span $H$, so this would imply that $H \subseteq \beta^\perp$, and we showed that $H$ is not contained in any $\beta^\perp$. □

Let $z_j$ be the $\exp(2\pi i/m_j)$ eigenvector for $c$ acting on $V \otimes \mathbb{C}$. Recall that, for any basis of $V$, the Jacobian $\det\left(\frac{\partial f_i}{\partial z_j}\right)$ is a nonzero multiple of $\Delta = \prod_{\beta \in \Phi^+} \beta$. We thus see that the determinant $\det\left(\frac{\partial f_i}{\partial z_j}\right)$ is nonzero at the point $z_1$.

Thus, after reordering, we may assume that

$$\left.\frac{\partial f_j}{\partial z_j}\right|_{z_1} \neq 0.$$

This means that $f_j$ must contain a monomial of the form $z_1^m z_j$. (In coordinates, we are differentiating with respect to $z_j$ and then plugging in $(1, 0, 0, \ldots, 0)$.) Since $\deg f_j = d_j$, this monomial must be $z_1^{d_j - 1} z_j$.

Thus, $z_1^{d_j-1} z_j$ is $c$ invariant and we deduce that $d_j - 1 + m_j \equiv 0 \bmod h$ or $d_j - 1 \equiv h - m_j \bmod h$.

Now, since $c$ is a real matrix, each $\exp(2\pi i m/h)$ eigenvalue is paired with an $\exp(2\pi i(h - m)/h)$ eigenvalue. So, reordering our indexing, we have $d_j - 1 \equiv m_j \bmod h$. We want to lift this congruence up to an equality.

Since $m_j < h$ and $d_j - 1 \equiv m_j \bmod h$, we have $d_j - 1 \ge m_j$. So $|T| = \sum d_j - 1 \ge \sum m_j$. Since the $m_j$ come in pairs summing to $h$, the sum on the right hand side is $(n/2)h$, which we also just showed equals $|T|$.