A LEMMA ABOUT THE JACOBIAN DETERMINANT

The point of this note is to prove the following lemma: Let k be a field of characteristic zero. Let $f_1, f_2, \ldots, f_n \in k[x_1, \ldots, x_n]$. Then $f_1, \ldots, f_n$ are algebraically dependent if and only if

$$\det \left[\frac{\partial f_i}{\partial x_j} \right] = 0.$$

This proof is taken from Humphreys *Reflection Groups and Coxeter Groups* Chapter 3.10. I would be interested in seeing better motivated, similarly elementary proofs.

We put $J = [\frac{\partial f_i}{\partial x_j}]$.

From a geometric perspective taking $k = \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_n(x))$ gives a map $F : \mathbb{R}^n \to \mathbb{R}^n$. The condition that $\det J = 0$ means that DF is everywhere singular so, by Sard's Theorem, $F(\mathbb{R}^n)$ has measure 0. This perhaps makes it plausible that $F(\mathbb{R}^n)$ lies in some algebraic hypersurface.

The result is false in characteristic p; take $n = 1$ and $f_1 = x_1^p$ for a counterexample.

Proof. First, suppose that $f_1, \ldots, f_n$ are algebraically dependent. Then $H(f_1, \ldots, f_n) = 0$ for some nonzero polynomial $H(y_1, \ldots, y_n)$. Choose such H of minimal degree. Since we are in characteristic 0, at least one derivative $\partial H/\partial y_i$ is nonzero and, since we chose H minimal, $\frac{\partial H}{\partial y_i}(f_1, \ldots, f_n)$ is nonzero. Put $h_i = \frac{\partial H}{\partial y_i}(f_1, \ldots, f_n)$ and $\vec{h} = [h_1\ h_2\ \cdots\ h_n]$. Then the chain rule says that $\vec{h}J$ is the vector of derivatives $\partial H(f_1, \ldots, f_n)/\partial x_i$, which is 0. So $\vec{h}$ is in the left kernel of $\mathrm{Ker}(J)$ and $\det J = 0$.

Now, suppose that $f_1, \ldots, f_n$ are algebraically independent. Any $n+1$ polynomials in n variables are algebraically dependent so, for each x_k, there is a polynomial relation

$$g_k(f_1, \ldots, f_n, x_jk = 0$$

for some polynomial $g_k(y_1, \ldots, y_n, z)$. We choose g_k of minimal z degree and, since $f_1, \ldots, f_n$ are algebraically independent, this minimal degree is not 0. The chain rule gives

$$\sum_j \left. \frac{\partial g_k}{\partial y_j} \right|_{(y_1, \ldots, y_n, z) = (f_1, \ldots, f_n, x_k)} \frac{\partial y_j}{\partial x_i} + \left. \frac{\partial g_k}{\partial z} \right|_{(y_1, \ldots, y_n, z) = (f_1, \ldots, f_n, x_k)} = 0. \qquad (*)$$

Put $K = [\frac{\partial g_k}{\partial y_j}|_{(y_1, \ldots, y_n, z) = (f_1, \ldots, f_n, x_k)}]$. Put $q_k = \frac{\partial g_k}{\partial z}|_{(y_1, \ldots, y_n, z) = (f_1, \ldots, f_n, x_k)}$. So we can rewrite equation $(*)$ as

$$JK = -\mathrm{diag}(q_1, q_2, \ldots, q_n) \qquad (\dagger).$$

Since g_k has positive z-degree, and we are in characteristic 0, we know $\frac{\partial g_k}{\partial z} \neq 0$. By the minimality of g_k, it stays nonzero when we plug in $(f_1, \ldots, f_n, x_k)$. So $q_k \neq 0$. So the right hand side of $(\dagger)$ has determinant $(-1) \prod q_k \neq 0$. This shows that $\det J \neq 0$.

$\square$