Problem Set 1: Due Friday, September 15

You should be able to begin work on all of these problems immediately!
See the course website for homework policy.

1. (a) The B_n hyperplane arrangement consists of the following list of hyperplanes in $\mathbb{R}^n$: $x_i \pm x_j = 0$ for $1 \leq i < j \leq n$ and $x_i = 0$ for $1 \leq i \leq n$. Show that the complement of these hyperplanes has $2^n n!$ connected components.

   (b) The D_n hyperplane arrangement is the subset of the B_n arrangement consisting of the hyperplanes $x_i \pm x_j = 0$ for $1 \leq i < j \leq n$. How many regions does the complement of the D_n arrangement have?

2. We recall/preview the following definitions from class: Let Φ be a finite collection of vectors in $\mathbb{R}^n$, such that $\alpha \in \Phi$ implies $-\alpha \in \Phi$. Let $\rho \in \mathbb{R}^n$ such that $\langle \alpha, \rho \rangle \neq 0$ for any $\alpha \in \Phi$. We define the set of ***positive roots***, Φ^+ to be those roots $\alpha \in \Phi$ with $\langle \alpha, \rho \rangle > 0$. We define a positive root to be ***simple*** if it is not a positive linear combination of other positive roots.

   In the following cases (which are known as B_n, D_n and F_4), describe the positive roots and the simple roots. We write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{R}^n$.

   (a) Φ is all vectors in $\mathbb{R}^n$ of the forms $\pm e_i \pm e_j$ (with $i \neq j$) and $\pm e_i$. Take $\rho = (1, 2, 3, \ldots, n)$.

   (b) Φ is all vectors in $\mathbb{R}^n$ of the forms $\pm e_i \pm e_j$ (with $i \neq j$). Take $\rho = (1, 2, 3, \ldots, n)$.

   (c) Φ is all vectors in $\mathbb{R}^4$ of the forms $\pm e_i \pm e_j$ (with $i \neq j$), $\pm e_i$ and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. Take $\rho = (1, 2, 4, 8)$.

3. Let V be a two dimensional real vector space with basis α_1, α_2 and let $\alpha_1^\vee$ and $\alpha_2^\vee \in V^\vee$ be the vectors such that $\langle \alpha_1^\vee, \alpha_1 \rangle = \langle \alpha_2^\vee, \alpha_2 \rangle = 2$ and $\langle \alpha_1^\vee, \alpha_2 \rangle = \langle \alpha_2^\vee, \alpha_1 \rangle = -2$. Let s_i act on V by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$ (note that $s_1^2 = s_2^2 = 1$). Let W be the group generated by s_1 and s_2.

   (a) Give a simple description of the orbits of α_1 and α_2 under W.

   (b) Let $D = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0\}$. Draw and label D, $s_1 D$, $s_2 D$, $s1s2D$, $s_2 s_1 D$.

   (c) Give a simple description of $\bigcup_{w \in W} wD$.

   (d) Now suppose that $\langle \alpha_1^\vee, \alpha_2 \rangle = \langle \alpha_2^\vee, \alpha_1 \rangle = -3$ instead of -2. Repeat parts (a), (b) and (c) with this change. Hint: Fibonacci numbers should occur.

4. This problem introduces the affine symmetric group $\tilde{A}_{n-1}$, which will be an important example of an infinite Coxeter group.

   Fix an integer $n \geq 3$. Define $\tilde{S}_n$ to be the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ which obey $w(i+n) = w(i) + n$, made into a group under composition. For $1 \leq i \leq n$, define the element $s_i \in \tilde{S}_n$ by

$$s_i(x) = \begin{cases} x+1 & x \equiv i \bmod n \\ x-1 & x \equiv i+1 \bmod n \\ x & \text{otherwise} \end{cases}.$$

   Define $\tilde{A}_{n-1}$ be $\{w \in \tilde{S}_n : \sum_{i=1}^n w(i) = \sum_{i=1}^n i\}$.

   (a) What is the order of $s_i s_j$?

   (b) Show that $\tilde{A}_{n-1}$ is a subgroup and is generated by the s_i.