Problem Set 10: Due Friday, December 8

See the course website for homework policy. This is the last problem set!

1. Let W be a finite Coxeter group and let V, R, S, S_+ and Δ have their usual meanings. Let $A = R/RS_+$ be the coinvariant algebra.
   (a) We define the ***socle*** of A to be $\{a \in A : va = 0 \ \forall v \in V\}$. Show that the socle of A is one dimensional, spanned by Δ. (Hint: Let a lie in the socle, and consider $\partial_\beta(\beta a)$.)
   (b) For any integer i between 0 and $\ell(w_0)$, multiplication defines a bilinear map $A_i \times A_{\ell(w_0)-i} \to A_{\ell(w_0)} \cong \mathbb{R}$. Show that this is a perfect pairing.

2. Let s_1, s_2, $\dots$, s_{n-1} be the standard Coxeter generators of S_n. As we will discuss in class, a ***Coxeter element*** is a permutation c with a reduced word of the form $c = s_{i_1} s_{i_2} \cdots s_{i_{n-1}}$, where $(i_1, i_2, \dots, i_{n-1})$ is a permutation of $(1, 2, \dots, n-1)$.
   (a) Show that c is an n-cycle $(a_1 a_2 \cdots a_n)$ and describe how to read off $(a_1 a_2 \cdots a_n)$ from $(i_1, i_2, \dots, i_{n-1})$.
   (b) Let $\vec{v} = (v_1, \dots, v_n) \in \mathbb{C}^n$ be a nonzero $e^{2\pi i/n}$ eigenvector of c. Describe the geometry of the n points v_1, v_2, $\dots$, $v_n \in \mathbb{C}$.
   (c) We focus on the particular case $c = s_1 s_3 s_5 \cdots s_2 s_4 s_6 \cdots$. Let $H \subset \mathbb{R}^n$ be the 2-plane of vectors of the form $(\Re(\alpha v_1), \Re(\alpha v_2), \dots, \Re(\alpha v_n))$ for $\alpha \in \mathbb{C}$. Describe the intersection of H with the fundamental domain $D = \{(x_1, \dots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n\}$.

3. Let Γ be a graph with n vertices. An ***orientation*** $\mathcal{O}$ of Γ is an assignment of a direction $i \to j$ to each edge (i, j) of Γ. A ***sink*** of $(\Gamma, \mathcal{O})$ is a vertex i where all adjacent edges are directed into i, and a ***sink*** is a vertex where all adjacent vertices are directed out. A ***sink-source reversal*** takes a sink and reverses all edges incident to it, making a source. Given a cycle $\gamma = (v_1, v_2, \dots, v_k)$ in Γ and an orientation $\mathcal{O}$ of Γ, the flow of $\mathcal{O}$ along Γ is

$$\mathrm{Flow}(\gamma, \mathcal{O}) = \#\{i : v_i \to v_{i+1}\} - \#\{i : v_i \leftarrow v_{i+1}\}$$

   where the indices are cyclic modulo k. An orientation $\mathcal{O}$ is called ***acyclic*** if $-k < \mathrm{Flow}(\gamma, \mathcal{O}) < k$ for every length k cycle γ in Γ.

   The aim of this problem is to prove the following result: If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two acyclic orientation of Γ, and $\mathrm{Flow}(\gamma, \mathcal{O}_1) = \mathrm{Flow}(\gamma, \mathcal{O}_2)$ for all cycles γ, then we can transform $\mathcal{O}_1$ to $\mathcal{O}_2$ by a sequence of sink-source reversals. We write $i \to_1 j$ for the orientation $\mathcal{O}_1$ and similarly for $\mathcal{O}_2$. With all that as prelude, we begin:
   (a) Prove the converse: If $\mathcal{O}_1$ can be transformed into $\mathcal{O}_2$ by a sequence of sink-source reversals, then $\mathrm{Flow}(\gamma, \mathcal{O}_1) = \mathrm{Flow}(\gamma, \mathcal{O}_2)$ for all cycles γ.
   We may, and do, reduce to the case that Γ is connected.
   (b) Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two acyclic orientation of Γ. Suppose that $\mathrm{Flow}(\gamma, \mathcal{O}_1) = \mathrm{Flow}(\gamma, \mathcal{O}_2)$ for all cycles γ. Show that there is a unique function h from the vertices of Γ to $\mathbb{Z}_{\geq 0}$ such that

1. For every edge (i, j) of Γ,

$$h(i) - h(j) = \begin{cases} 1 & i \leftarrow_1 j \text{ and } i \rightarrow_2 j \\ 0 & (i \rightarrow_1 j \text{ and } i \rightarrow_2 j) \text{ or } (i \leftarrow_1 j \text{ and } i \leftarrow_2 j) \\ -1 & i \rightarrow_1 j \text{ and } i \leftarrow_2 j \end{cases}.$$

2. For every vertex v of Γ, we have $h(v) \geq 0$ and, for at least one vertex, we have $h(v) = 0$.

We define $d(\mathcal{O}_1, \mathcal{O}_2) = \sum_{v \in \Gamma} h(v)$. Our proof is by induction on d.

(c) Explain why the base case, $d(\mathcal{O}_1, \mathcal{O}_2) = 0$, holds.

(d) Let $d > 0$ and set $H = \max_v h(v)$. Show that there is some v_0 with $h(v) = H$ which is a sink of $\mathcal{O}_1$.

(e) Let $\mathcal{O}_1'$ be the orientation obtained from $\mathcal{O}_1$ by a sink-source reversal at v_0. Show that $d(\mathcal{O}_1', \mathcal{O}_2) = d(\mathcal{O}_1, \mathcal{O}_2) - 1$.