Problem Set 2: Due Monday, September 25

See the course website for homework policy.

1. Let W be the subgroup of $\mathrm{GL}_3(\mathbb{R})$ generated by the reflections

$$s_1 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad s_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Let $D = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0\}$.

   (a) Compute the rays of the (polyhedral) cones D, $s_1 D$, $s_1 s_2 D$.

   (b) Show that W preserves the (round) cone $C = \{(x_1 : x_2 : x_3) : x_1 x_2 + x_1 x_3 + x_2 x_3 \geq 0\}$.

   (c) Sketch how D, $s_1 D$ and $s_1 s_2 D$ lie in C.

2. For $1 \leq i < j \leq n$, let $m_{ij} \in \{2, 3, \ldots, \infty\}$. Let W be the group generated by s_1, s_2, …, s_n subject to relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$. Define a graph Γ_2 with vertex set 1, 2, …, n and an edge (i, j) if m_{ij} is odd, where ∞ is considered even.

   (a) Show that the number of homomorphisms $W \to \{\pm 1\}$ is $2^{\#(\text{connected components of } \Gamma_2)}$.

   (b) Show that s_i and s_j are conjugate in W if and only if i and j are in the same connected component of Γ_2

3. Let V be a vector space with basis α_1, …, α_n. Let A be an $n \times n$ matrix with $A_{ii} = 2$ and $A_{ij} = 0$ if and only if $A_{ji} = 0$. Let $\alpha_i^\vee$ in $V^\vee$ be the vector with $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$. Define $s_i : V \to V$ by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. Let W be the group generated by the s_i.

   (a) Suppose that there is symmetric bilinear form $(\ ,\ )$ on V which is invariant under W, and where $(\alpha_i, \alpha_i) \neq 0$. Defining $d_i = (\alpha_i, \alpha_i)$, show that $d_i A_{ij} = d_j A_{ji}$.

   (b) Conversely, suppose that there exist positive real numbers d_i such that $d_i A_{ij} = d_j A_{ji}$. Show that we can define a symmetric bilinear form $(\ ,\ )$ on V with $(\alpha_i, \alpha_i) = d_i$ such that s_i is the orthogonal reflection in α_i.

4. Let $n \geq 3$ be a positive integer. Let p be a real number larger than 1. Let V be the vector space of sequences $(a_i)_{i \in \mathbb{Z}}$ such that $a_{i+n} = p a_i$. Let the group $\tilde{A}_{n-1}$ (see previous problem set) act on V by $w(a)_i = a_{w^{-1}(i)}$. The funny inverse is to make it a left action.

   (a) Choose a basis for V, and write the matrices of s_1, s_2, …, s_n in your basis.

   (b) Give explicit vectors $\alpha_i \in V$ and $\alpha_i^\vee \in V^\vee$ such that $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. Choose your signs such that $\langle \alpha_i^\vee, \ \rangle$ is positive on the sequence $(p^{i/n})$.

   (c) Compute the Cartan matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.
   Let $D = \{x \in V : \langle \alpha_i^\vee, x \rangle \geq 0,\ 1 \leq i \leq n\}$. Let $T = \{(x_i) \in V : x_i > 0\}$.

   (d) For $n = 3$, sketch the intersections of T, D and the hyperplanes $\alpha_i^\perp$, with the unit sphere.

   (e) Prove that $\bigcup_{w \in \tilde{A}_{n-1}} wD = T \cup \{0\}$.