Problem Set 3: Due Friday, September 29

See the course website for homework policy.

1. Let V be a finite dimensional real vector space equipped with a positive definite symmetric bilinear form $\cdot$. Let Λ be a discrete additive subgroup of V with $\mathrm{Span}_{\mathbb{R}}(\Lambda) = V$. Define G to be the group of linear transformations $g : V \to V$ with $g(u) \cdot g(v) = u \cdot v$ and $g(\Lambda) = \Lambda$.
   (a) Show that G is finite.
   (b) Show that, for $g \in G$, we have $\mathrm{Tr}\, g \in \mathbb{Z}$.
   (c) Let $\dim V = 2$ and let $g \in G$. Show that g has order 1, 2, 3, 4 or 6.

2. This problem will explore a representation where the α_i and $\alpha_i^\vee$ are not linearly independent. V and $V^\vee$ be 3 dimensional, written as column and row vectors respectfully, and take

$$\alpha_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \alpha_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} \quad \alpha_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \alpha_4 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$
$$\alpha_1^\vee = \begin{bmatrix}2 & 0 & 0\end{bmatrix} \quad \alpha_2^\vee = \begin{bmatrix}-2 & 0 & 0\end{bmatrix} \quad \alpha_3^\vee = \begin{bmatrix}0 & 2 & 0\end{bmatrix} \quad \alpha_4^\vee = \begin{bmatrix}0 & -2 & 0\end{bmatrix}$$

   We define $D = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0\}$. Recall that s_i acts on $V^\vee$ by $s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i^\vee$.
   (a) Show that the matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is a Cartan matrix. What are the m_{ij}?
   (b) Let $V_1^\vee$ be the affine linear space $\left\{ \left[\begin{smallmatrix} x \\ y \\ 1 \end{smallmatrix} \right] \right\}$ in $V_1^\vee$. Show that W preserves $V_1^\vee$.
   (c) In terms of the coordinates (x, y) on $V_1^\vee$, write down the action of the s_i on $V_1^\vee$. Give inequalities on x and y describing $D_1 := D \cap V_1^\vee$.
   (d) Draw and label the domains wD_1 in the two dimensional plane $V_1^\vee$ for several values of w.

3. This problem describes a different representation of $\tilde{A}_{n-1}$ from the one on Problem Set 2.

   Let $n \geq 3$ be a positive integer. Let V be the vector space of sequences $(a_i)_{i \in \mathbb{Z}}$ such that $a_{i+n} - a_i$ is a constant independent of i. Let $\tilde{A}_{n-1}$ act on V by $w(a)_i = a_{w^{-1}(i)}$.
   (a) Choose a basis for V, and write the matrices of $s_1, s_2, \ldots, s_n$ in your basis.
   (b) Give explicit vectors $\alpha_i \in V$ and $\alpha_i^\vee \in V^\vee$ such that $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. Choose your signs such that $\langle \alpha_i^\vee, \ \rangle$ is positive on the point $x_i = i$.
   (c) Compute the Cartan matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.
   Once again, let $D = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0, \ 1 \leq i \leq n\}$.
   Let $\bar{V}$ be the quotient of V by the vector space of constant sequences. Let $\bar{V}_1$ be the affine subspace $a_{i+n} = a_i + 1$ of $\bar{V}$. Note that $\dim \bar{V}_1 = n - 1$, which means we can draw it for $n = 3$. I'll write $\bar{D}$ for the image of D in $\bar{V}$ and $\bar{D}_1$ for the intersection $\bar{D} \cap \bar{V}_1$.
   (d) For $n = 3$, draw $\bar{D}_1$, $s_1\bar{D}_1$, $s_2\bar{D}_1$, $s_3\bar{D}_1$, $s1s2\bar{D}_1$, $s_2s_1\bar{D}_1$ and $s_1s_2s_1\bar{D}_1$ inside $\bar{V}_1$. Draw the hyperplanes fixed by s_1, s_2, s_3.
   (e) Show that $\bar{V}_1 = \bigcup_{w \in \tilde{A}_{3-1}} w\bar{D}_1$.