Problem Set 4: Due Friday, October 6

See the course website for homework policy.

1. (a) Describe the longest element w_0, and its action on V, when W each of A_n, B_n and D_n.
   (b) In which cases does w_0 act on V by $-\mathrm{Id}$?

2. In this problem, we will show that T "is the set of reflections of W". We define W, s_1, …, s_n, α_1, …, α_n and $\alpha_1^\vee$, …, $\alpha_n^\vee$ as usual, including the condition that $D \neq \emptyset$. We define $T = \{ws_iw^{-1} : w \in W, 1 \leq i \leq n\}$. Let $t \in W$ act on $V^\vee$ by a reflection, meaning that t fixes a codimension 1 subspace H and acts by -1 on $V^\vee/H$. We will show that $t \in T$.

   Let $t = s_{i_1}s_{i_2}\cdots s_{i_\ell}$. Put $v_k = s_{i_1}s_{i_2}\cdots s_{i_k}$.
   (a) Show that H does not pass through D° (remember, we don't know H is of the form $\beta^\perp$ yet). Deduce that D and tD are on opposite sides of H.
   (b) Show that H does not pass through any of the v_kD°.
   (c) Show that H is the wall along which $v_{k-1}D$ borders v_kD for some k.
   (d) Deduce that $t \in T$.

3. Let u and $v \in W$. The point of this problem is to show that $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$ if and only if there is a reduced word $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ for v such that $u = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some $k \leq \ell$.
   (a) Show that, if $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ for v such that $u = s_{i_1}s_{i_2}\cdots s_{i_k}$, then $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$.
   (b) Show the reverse implication. Hint: Recall the following lemma from class: If s_i is a left ascent of w, then $\mathrm{inv}(s_iw) = s_i(\mathrm{inv}(w) \setminus \{s_i\})s_i$.

4. Let W be a Coxeter group with s_i and m_{ij} as usual. Given a word in W, the (i,j) ***braid move*** is to replace the substring $\overbrace{s_is_js_is_j\cdots}^{\text{length } m_{ij}}$ by $\overbrace{s_js_is_js_i\cdots}^{\text{length } m_{ij}}$. Let $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ and $s_{j_1}s_{j_2}\cdots s_{j_\ell}$ be two reduced words with the same product w. The aim of this problem is to show that we can transform $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ to $s_{j_1}s_{j_2}\cdots s_{j_\ell}$ using only braid moves.

   Our proof is by induction on ℓ, so assume we have proven the result for any pair of words of shorter length.
   (a) If $i_1 = j_1$, show we are done.
   Suppose from now on that $i_1 \neq j_1$. We abbreviate $i_1 = i$, $j_1 = j$ and $m_{ij} = m$.
   (b) Show that all m reflections in $\langle s_i, s_j \rangle$ are inversions of w. (Hint: Geometry!)
   (c) Show that there is a reduced word for w of the form $\overbrace{s_is_js_is_j\cdots}^{\text{length } m}s_{k_{m+1}}s_{k_{m+2}}\cdots s_{k_\ell}$.
   (d) Conclude the proof.