Problem Set 5: Due Friday, October 13

See the course website for homework policy.

1. Let $t \in T$ and let β be the corresponding positive root. Let $w \in W$. Check that $t \in \mathrm{inv}(w)$ if and only if $w^{-1}\beta$ is a negative root.

2. Let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a word in W and set $\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}$ and let $t_k = s_{i_1} s_{i_2} \cdots s_{i_k} \cdots s_{i_2} s_{i_1}$ be the reflection over β_k. Fix $t \in T$ and let $k_1, k_2, \ldots k_r$ be the indices k for which $t = t_k$. Show that β_{k_j} is a positive or negative root according to whether j is odd or even. In particular, $s_{i_1} \cdots s_{i_\ell}$ is reduced if and only if all β_k are positive roots.

3. (For those who have seen Catalan numbers somewhere.) Let Π be the root poset of A_n, so the elements of Π are Φ^+ and $\beta_1 \geq \beta_2$ if $\beta_1 = \beta_2 + \sum c_i \alpha_i$ with $c_i \geq 0$. Recall that $I \subseteq \Pi$ is called an order ideal if, whenever $\beta_1 \in I$ and $\beta_1 \geq \beta_2$, we have $\beta_2 \in I$. Show that the number of order ideals in Π is the Catalan number $\frac{1}{2n+3}\binom{2n+3}{n+1}$.

4. This problem is some geometric computations to help us understand the geometry of D in the finite case, but I'll write it just as a geometry problem. Let V be a vector space equipped with a positive definite symmetric bilinear form. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a basis for $\mathbb{R}$ such that $\alpha_i \cdot \alpha_j \leq 0$ for $i \neq j$. Let $D = \{x \in V : \alpha_i \cdot x \geq 0\}$.
   (a) Let $\gamma = \sum c_i \alpha_i \in D \setminus \{0\}$. Show that all the c_i are positive. (Hint: Write $\gamma = \gamma_+ - \gamma_-$ where $\gamma_+ = \sum_{c_i > 0} c_i \alpha_i$ and $\gamma_- = \sum_{c_i < 0} (-c_i)\alpha_i$.) Compute $\gamma \cdot \gamma_-$ in two ways.)
   (b) For any γ_1 and γ_2 in $D \setminus \{0\}$, show that $\gamma_1 \cdot \gamma_2 > 0$.

5. Make the usual definitions. We have seen that W acts freely on $\bigcup_{w \in W} wD^\circ$. The aim of this problem is to investigate what happens on the boundary of D.

   Let $x \in D$ and let $I = \{i : \langle \alpha_i, x \rangle = 0\}$. Let W_I be the subgroup $\langle s_i \rangle_{i \in I} \subset W$; let $\mathrm{Stab}_x = \{w \in W : w(x) = x\}$ and let $Q(x) = \{w \in W : w(x) \in D\}$. In this problem, we will show that $W_I = \mathrm{Stab}_x = Q(x)$.
   (a) Show that $W_I \subseteq \mathrm{Stab}_x(W) \subseteq Q(x)$. Show that $\mathrm{Stab}_x(W)$ and $Q(x)$ are unions of cosets wW_I.
   (b) Let $j \notin I$ and $w \in Q(x)$. Show that s_j is **not** a right descent of w.
   (c) Show that $W_I = \mathrm{Stab}_x(W) = Q(x)$.
   (d) Show that, for any $x \in \mathrm{Tits}(W)$, there is precisely one point in $Wx \cap D$.
   (e) Show that, for any $x \in \mathrm{Tits}(W)$, the stabilizer $\mathrm{Stab}_x(W)$ is of the form uW_Iu^{-1} for some u and some I. A subgroup of the form uW_Iu^{-1} is called a ***parabolic subgroup***.