Problem Set 6: Due Friday, October 27

See the course website for homework policy.

Note: Because I expect the course to be providing an overview of Lie groups at this point, the problems are not closely tied to the current material. They are either things that I think it would be nice for you to know, or things which will be useful when we start studying invariant theory.

1. Let Φ be a finite root system with simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$.
   (a) Let s_i be a simple generator and β a positive root other than α_i. Show that $s_i\beta$ is a **positive** root. (Hint: Quote an earlier problem.)
   Define $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$.
   (b) Show that $\langle \alpha_i^\vee, \rho\rangle = 1$ for $1 \leq i \leq n$.
   (c) Take our usual models of the finite classical crystallographic root systems: $\{e_i - e_j\}$ in type A, $\{\pm e_i \pm e_j,\ e_k\}$ in type B, $\{\pm e_i \pm e_j,\ 2e_k\}$ in type C and $\{\pm e_i \pm e_j\}$ in type D and compute ρ in these cases.
   For a reflection $t \in T$, let β_t be the positive root with $t\beta_t = -\beta_t$.
   (d) For any $w \in W$, show that $w(\rho) = \rho - \sum_{t\in \mathrm{inv}(w)} \beta_t$

2. This problem describes a model for the group $\tilde{C}_n$ similar to our model for $\tilde{A}_{n-1}$ in terms of affine permutations. We define $\tilde{C}_n$ by its Coxeter diagram:

$$s_0 \overset{4}{\text{———}} s_1 \text{———} \cdots \text{———} s_{n-1} \overset{4}{\text{———}} s_n$$

   Let $Z = \{k + 1/2 : k \in \mathbb{Z}\}$. Let $\tilde{S}_n^C$ be the set of bijections $w : Z \to Z$ such that

$$w(z + 2n) = w(z) + 2n \text{ and } w(-z) = -w(z).$$

   Our goal will be to show $\tilde{S}_n^C \cong \tilde{C}_n$.
   (a) Describe a map $\tilde{C}_n \to \tilde{S}_n^C$ by describing where to send the generators $s_0, s_1, \ldots, s_n$.
   Let $V^\vee$ be the vector space of functions $a : Z \to \mathbb{R}$ obeying $a(-z) = -a(z)$ and such that $a(z + 2n) - a(z)$ is a constant independent of z.
   (b) Choose a basis for $V^\vee$ and write down the action of $s_0, \ldots, s_n$ in that basis. Write $s_i(x)$ in the form $x - \langle \alpha_i, x\rangle\alpha_i^\vee$ for some roots α_i and $\alpha_i^\vee$.
   (c) Check that the α_i and $\alpha_i^\vee$ pair by the Cartan matrix for $\tilde{C}_n$.
   (d) Show that $\tilde{C}_n \to \tilde{S}_n^C$ is an isomorphism.

3. Let W be a Coxeter group with generators s_1, s_2, $\ldots$, s_n. For $I \subseteq [n]$, we defined W_I to be the subgroup generated by $\{s_i : i \in I\}$. We define ${}^I W$ and W^I by

$$ {}^I W = \{w \in W : s_i \text{ is a left ascent of } w \text{ for all } i \in I\}$$

$$W^I = \{w \in W : s_i \text{ is a right ascent of } w \text{ for all } i \in I\}.$$

   (a) Show that each $w \in W$ can be factored as uv, with $u \in W_I$ and $v \in {}^I W$.

   (b) Show that we have $v \in {}^I W$ if and only if $\mathrm{inv}(v) \cap W_I = \emptyset$.

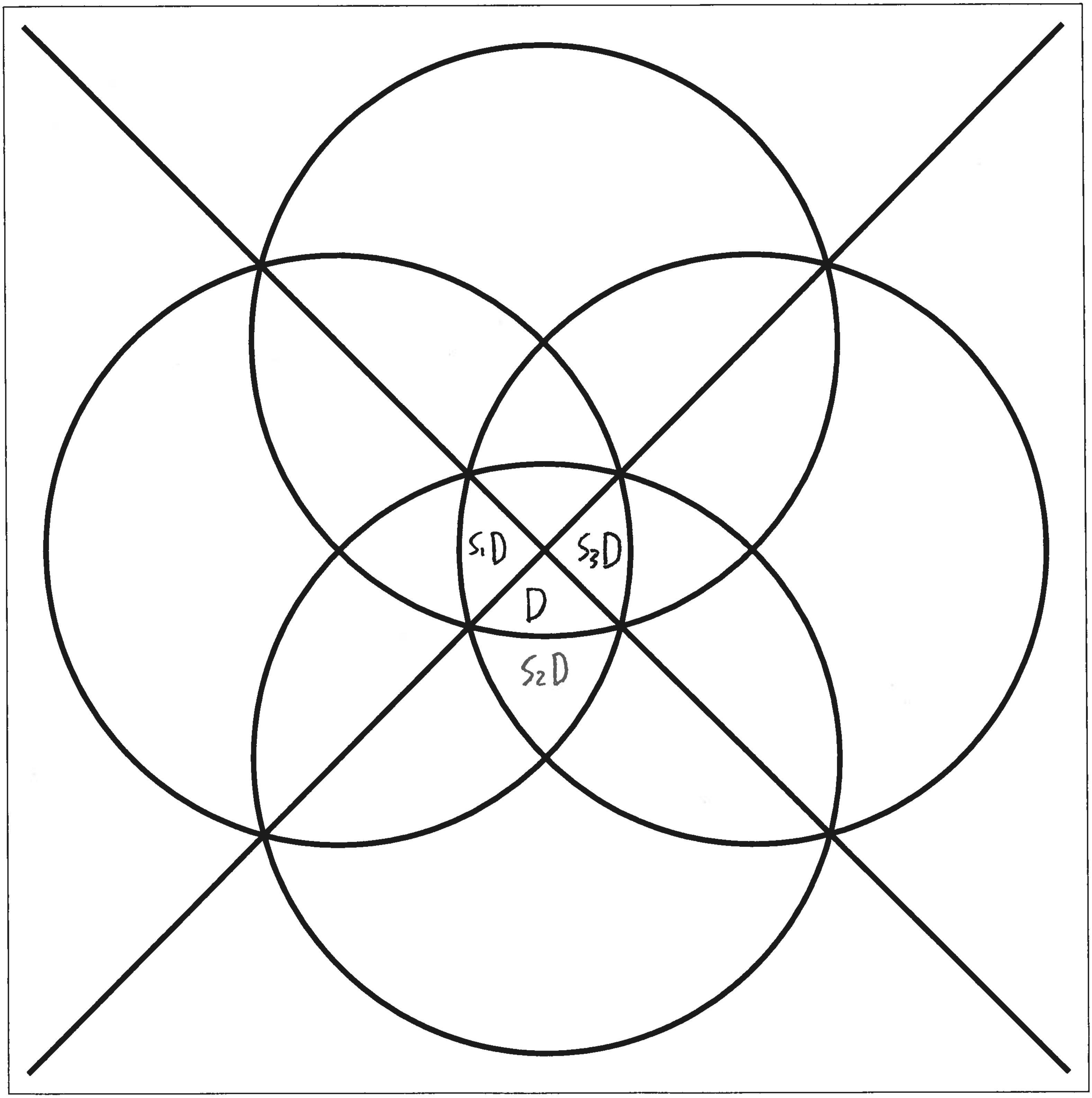
   (c) Suppose that $w = uv$ with $u \in W_I$ and $v \in {}^I W$. Show that $\mathrm{inv}(u) = \mathrm{inv}(w) \cap W_I$.

   (d) Let $w \in W$. Show that there is a unique factorization of w as $w = uv$ with $u \in W_I$ and $v \in {}^I W$. The unique u and v are denoted w_I and ${}^I w$. We define ${}_I w$ and w^I similarly.

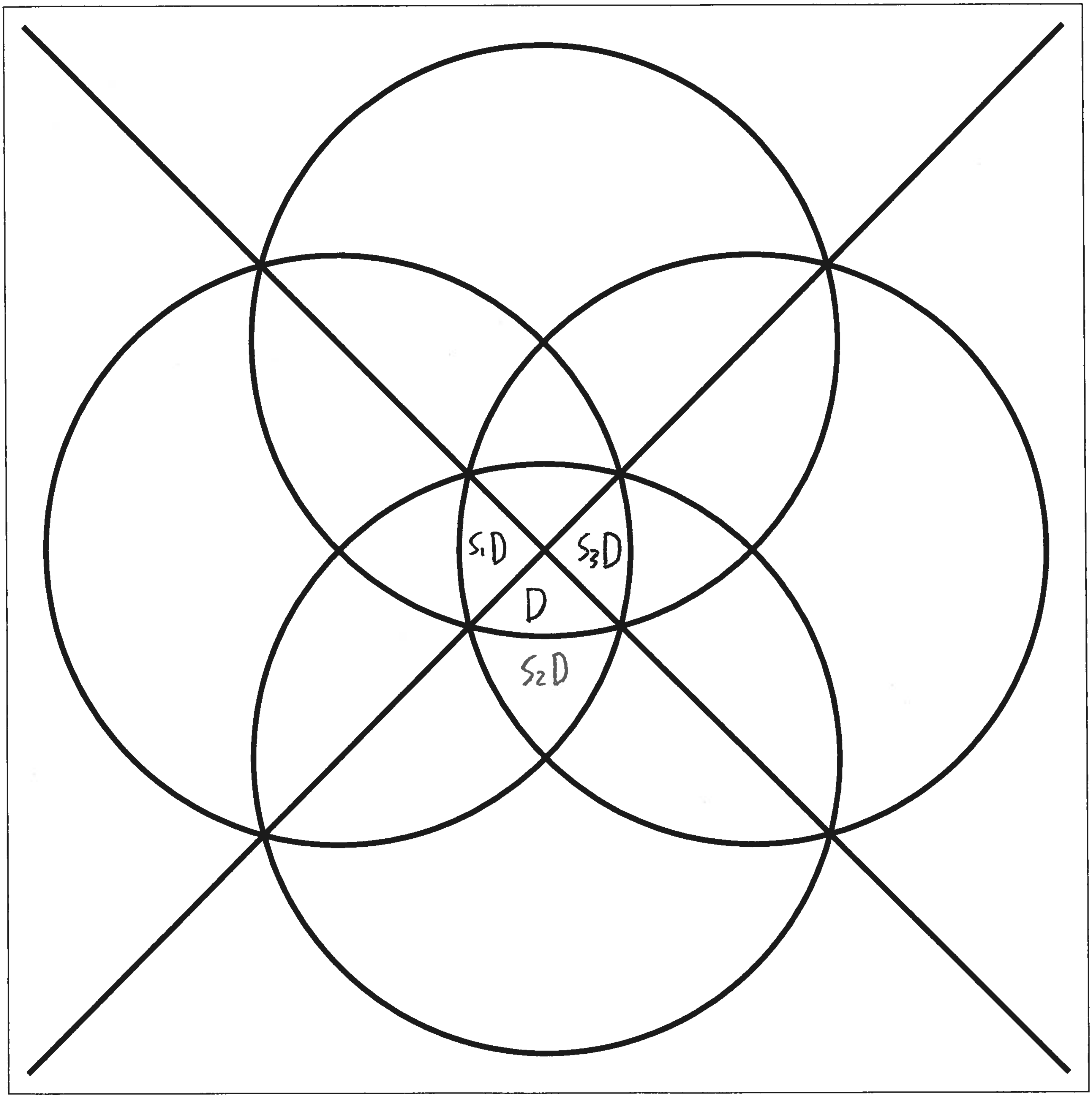
   (e) The next two pages of this problem set feature the hyperplane arrangement A_3 (drawn in stereographic projection) with regions D, $s_1 D$, $s_2 D$, $s_3 D$ labeled. Set $I = \{1, 3\}$.

   On the first page, use four colors to indicate which w's have w_I equal to 1, s_1, s_3 or $s_1 s3$.

   On the second page, outline the cosets uW_I and color according to the value of ${}_I w$.



S_1D
S_3D
D
S_2D


S1D
S3D
D
S2D