Problem Set 7: Due Friday, November 3

See the course website for homework policy.

1. Let W be a Coxeter group, let T be the set of reflections. For $X \subseteq T$, define
$$s_i * X = \begin{cases} s_i X s_i^{-1} \cup \{s_i\} & s_i \notin X \\ s_i X s_i^{-1} \setminus \{s_i\} & s_i \in X \end{cases}.$$
Show that this extends to an action of W on the set of subsets of T.

2. Let G be a finite group and let V be a real vector space on which G acts (by linear maps). We write ρ for the map $G \to \mathrm{GL}(V)$. Let V^G be the space of G-invariants and define
$$V_0 = \left\{ \vec{v} \in V : \sum_{g \in G} \rho(g)(\vec{v}) = 0 \right\}.$$
   (a) Show that $V = V^G \oplus V_0$.
   (b) Show that
$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} \rho(g).$$

3. Let V be an n-dimensional vector space. Let g be a linear endomorphism of g and let $\mathrm{Sym}^k(g)$ be the induced action of g on $\mathrm{Sym}^k V$. Show that
$$\sum_{k=0}^{\infty} t^k \operatorname{Tr} \mathrm{Sym}^k(g) = \frac{1}{\det(\mathrm{Id} - tg)}.$$

4. Let V be a finite dimensional real vector space and let $t : V \to V$ be a reflection with $t\beta = -\beta$ for some nonzero $\beta \in V$.
   (a) Let R be the symmetric algebra $\bigoplus \mathrm{Sym}^\bullet(V)$. Let $f \in R$ obey $t \cdot f = -f$. Show that β divides f in the ring R.
   (b) Let $\Lambda \subset V$ be a t-invariant lattice and suppose that $\mathbb{R}\beta \cap \Lambda = \mathbb{Z}\beta$. Let $\widehat{R}$ be the group ring $\mathbb{R}[\Lambda]$. Let $f \in \widehat{R}$ obey $t \cdot f = -f$. Show that $z^\beta - 1$ divides f in the ring $\widehat{R}$.

5. Let $I_2(m)$ act on $\mathbb{R}^2$ with the action generated the reflection $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and the rotation $\left[\begin{smallmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{smallmatrix}\right]$. We'll write (x, y) for the coordinates on $\mathbb{R}^2$.

   Find a degree m polynomial in x and y, invariant under $I_2(m)$, which is not in the subring generated by $x^2 + y^2$.