# Problem Set 8: Due Friday, November 10

See the course website for homework policy.

1. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n_{>0}$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, we adopt $x^a$ as shorthand for $\prod x_i^{a_i}$.
   (a) Let the Coxeter group $B_n$ act on $\mathbb{R}^n$ in the standard manner, and recall that we computed that $\rho = (1/2, 3/2, \dots, n - 1/2)$ in this case. Show that
$$\sum_{w \in B_n} (-1)^{\ell(w)} x^{w(\rho)} = \det \begin{bmatrix} x_1^{1/2} - x_1^{-1/2} & x_2^{1/2} - x_2^{-1/2} & \cdots & x_n^{1/2} - x_n^{-1/2} \\ x_1^{3/2} - x_1^{-3/2} & x_2^{3/2} - x_2^{-3/2} & \cdots & x_n^{3/2} - x_n^{-3/2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1/2} - x_1^{-n+1/2} & x_2^{n-1/2} - x_2^{-n+1/2} & \cdots & x_n^{n-1/2} - x_n^{-n+1/2} \end{bmatrix}$$
   (b) Let the Coxeter group $C_n$ act on $\mathbb{R}^n$ in the standard manner, and recall that we computed that $\rho = (1, 2, \dots, n)$ in this case. Show that
$$\sum_{w \in C_n} (-1)^{\ell(w)} x^{w(\rho)} = \det \begin{bmatrix} x_1^1 - x_1^{-1} & x_2^1 - x_2^{-1} & \cdots & x_n^1 - x_n^{-1} \\ x_1^2 - x_1^{-2} & x_2^2 - x_2^{-2} & \cdots & x_n^2 - x_n^{-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n - x_1^{-n} & x_2^n - x_2^{-n} & \cdots & x_n^n - x_n^{-n} \end{bmatrix}$$
   (c) Let the Coxeter group $D_n$ act on $\mathbb{R}^n$ in the standard manner, and recall that we computed that $\rho = (0, 1, 2, \dots, n-1)$ in this case. Show that
$$\sum_{w \in D_n} (-1)^{\ell(w)} x^{w(\rho)} = \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1^1 + x_1^{-1} & x_2^1 + x_2^{-1} & x_3^1 + x_3^{-1} & \cdots & x_n^1 + x_n^{-1} \\ x_1^2 + x_1^{-2} & x_2^2 + x_2^{-2} & x_3^2 + x_3^{-2} & \cdots & x_n^2 + x_n^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} + x_1^{1-n} & x_2^{n-1} + x_2^{1-n} & x_3^{n-1} + x_3^{1-n} & \cdots & x_n^{n-1} + x_n^{1-n} \end{bmatrix}$$

2. Let $W$ be a Coxeter group and $W_I$ a parabolic subgroup. We recall the minimal coset representatives $W^I$ from Problem Set 6, Problem 3. We put $[n] = \{1, 2, 3, \dots, n\}$.
   (a) Show that
$$\sum_{w \in W} q^{\ell(w)} = \left( \sum_{w \in W_I} q^{\ell(w)} \right) \left( \sum_{w \in W^I} q^{\ell(w)} \right).$$
   Take $W = S_n$ and $W_I = S_k \times S_{n-k}$ (in the obvious way).
   (b) Give a bijection between $W^I$ and the $k$-element subsets of $[n]$ such that, if $w \in W^I$ corresponds to $A \subset [n]$, then $\ell(w) = \#\{(a, b) \in A \times ([n] \setminus A) : a > b\}$.
   (c) Explain what formula you have proved for
$$\sum_{\substack{A \subset [n] \\ |A| = k}} q^{\#\{(a,b) \in A \times ([n] \setminus A) : a > b\}}.$$