Problem Set 9: Due Friday, November 17

See the course website for homework policy.

1. (a) Let R be a commutative ring, and $n \in R$ a nilpotent element, meaning $n^N = 0$ for some $N > 0$. Show that $1 + n$ is a unit.
   (b) Let R be a commutative ring, $\mathfrak{m}$ a maximal ideal and N a positive integer. Show that $R/\mathfrak{m}^N$ is a local ring, meaning that every element is either in $\mathfrak{m}$ or is a unit.

2. The aim of this question is to prove the following lemma: Let R be a noetherian ring with maximal ideal $\mathfrak{m}$. Let G be a group of automorphisms of R which take $\mathfrak{m}$ to itself. Let $S = R^G$ and let $\mathfrak{n} = S \cap \mathfrak{m}$. Then there is an integer N such that $\mathfrak{m}^N \subset R\mathfrak{n}$.
   (a) Show that, for any $x \in \mathfrak{m}$, we have $x^{|G|} \in R\mathfrak{n}$. (Hint: Consider $\prod_{g \in G}(x - g(x))$.)
   (b) Let $\mathfrak{m} = \langle x_1, x_2, \ldots, x_k \rangle$; since R is noetherian, this is a finite list. Show that, for any y_1, y_2, …, $y_{k|G|} \in \mathfrak{m}$, we have $\prod y_i \in R\mathfrak{n}$.

3. This question introduces the divided difference operators in type A. Define the operator ∂_i on $\mathbb{R}[x_1, \ldots, x_n]$ by
$$\partial_i(f) = \frac{f(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.$$
   (a) Show that, if $\partial_1 f = \partial_2 f = \cdots = \partial_{n-1} f = 0$, then f is symmetric.
   (b) Show that $\partial_i^2 f = 0$ for any $f \in \mathbb{R}[x_1, \ldots, x_n]$.
   (c) Compute all nonzero polynomials which can be obtained by repeatedly applying ∂_1 and ∂_2 to $x_1^2 x_2$.

4. In class, we showed how to compute $\sum_{w \in W} q^{\ell(w)}$ for W a finite crystallographic group. In this problem we will give a method which, in theory, computes $\sum_{w \in W} q^{\ell(w)}$ for any Coxeter group W. We recall the notations W_I and W^I from Problem Set 6, and the formula
$$\sum_{w \in W^I} q^{\ell(w)} = \frac{\sum_{w \in W} q^{\ell(w)}}{\sum_{w \in W_I} q^{\ell(w)}}$$
from Problem Set 8. We write s_1, s_2, …, s_n for the simple generators of W and $[n] = \{1, 2, \ldots, n\}$.
   (a) Show that, if W is finite, then the only element of W for which s_1, s_2, …, s_n are all right descents is w_0. Show that, if W is infinite, then no element has this property.
   (b) Show that
$$\sum_{I \subseteq [n]} (-1)^{n-|I|} \sum_{w \in W^I} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & W \text{ finite} \\ 0 & W \text{ infinite} \end{cases}.$$
   (c) Check that the above formula is correct for $I_2(m)$.
   (d) Use the above formula to compute $\sum_{w \in \tilde{A}_2} q^{\ell(w)}$. You may take as known that $\sum_{w \in A_2} q^{\ell(w)} = (1+q)(1+q+q^2)$ and similar formulas for the smaller parabolic subgroups.
   (e) Prove that, for any W, the formal sum $\sum_{w \in W} q^{\ell(w)}$ is a rational function of q.