The Weyl Denominator Identities for the Classical Types

Let (W, Φ) be a finite crystallographic Coxeter group and let $\rho = (1/2)\sum_{\beta\in\Phi^+}\beta$. Then the Weyl denominator identity is

$$\sum_{w\in W}(-1)^{\ell(w)}z^{w(\rho)} = \prod_{\beta\in\Phi^+}(z^{\beta/2} - z^{-\beta/2}).$$

In each of the classical types, this can be rewritten as a determinant. We state the results here. Table taken from Chapter 2.4 in *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture* by David Bressoud:

$$\boldsymbol{A_{n-1}}: \quad \det\begin{bmatrix} z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ z_1 & z_2 & \cdots & z_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \prod_{1\le i<j\le n}(z_i - z_j).$$

$$\boldsymbol{B_n}: \quad \det\begin{bmatrix} 1 - z_1^{2n-1} & 1 - z_2^{2n-1} & \cdots & 1 - z_n^{2n-1} \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{n-2} - z_1^{n+1} & z_2^{n-2} - z_2^{n+1} & \cdots & z_n^{n-2} - z_n^{n+1} \\ z_1^{n-1} - z_1^{n} & z_2^{n-1} - z_2^{n} & \cdots & z_n^{n-1} - z_n^{n} \end{bmatrix} = \prod_{1\le k\le n}(1 - z_k)\prod_{1\le i<j\le n}(z_i - z_j)(z_iz_j - 1).$$

$$\boldsymbol{C_n}: \quad \det\begin{bmatrix} 1 - z_1^{2n} & 1 - z_2^{2n} & \cdots & 1 - z_n^{2n} \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{n-2} - z_1^{n+2} & z_2^{n-2} - z_2^{n+2} & \cdots & z_n^{n-2} - z_n^{n+2} \\ z_1^{n-1} - z_1^{n+1} & z_2^{n-1} - z_2^{n+1} & \cdots & z_n^{n-1} - z_n^{n+1} \end{bmatrix} = \prod_{1\le k\ne n}(1 - z_k^2)\prod_{1\le i<j\le n}(z_i - z_j)(z_iz_j - 1).$$

$$\boldsymbol{D_n}: \quad \det\begin{bmatrix} 1 + z_1^{2n} & 1 + z_2^{2n} & \cdots & 1 + z_n^{2n} \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{n-1} + z_1^{n+1} & z_2^{n-1} + z_2^{n+1} & \cdots & z_n^{n-1} + z_n^{n+1} \\ z_1^{n} & z_2^{n} & \cdots & z_n^{n} \end{bmatrix} = \prod_{1\le i<j\le n}(z_i - z_j)(z_iz_j - 1).$$