POSET STRUCTURE OF COXETER GROUPS

These are class notes from Math 665, taught at the University of Michigan, Fall 2019. These notes pick up in the second month of the course; see the file "Basic Structure of Coxeter Groups" for material before that.

Class taught by David E Speyer; class notes edited by David and written by the students: Anna Brosowsky, Shelby Cox, Will Dana, Vignesh Jagathese, Zhi Jiang, Will Newman, Ruilin Shi and Danny Stoll for their comments and suggestions. Comments and corrections are welcome!

Contents

October 18 – Introduction to Weak Order

We started talking about weak order, including a review of some concepts that we had discussed before. We described the weak order for finite Coxeter groups of type A, B, C, D, and their affine counterparts $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$. At the end of class, we had time to discuss some basic poset vocabulary.

There are two types of weak order: right and left.

Definition. We define

- $u \leq w$ in ***right weak order*** if $l(w) = l(u) + l(u^{-1}w)$, or equivalently, $w = uv$ for some v, with $l(w) = l(u) + l(v)$.
- $u \leq w$ in ***left weak order*** if $l(w) = l(u) + l(wu^{-1})$, or equivalently, $w = vu$ for some v, with $l(w) = l(u) + l(v)$.

We denote right order by $\leq_R$ and left order by $\leq_L$. We will usually prefer right order.

It is easily checked that both left and right weak orders are partial orders.

Right and left weak orders are isomorphic via the map $w \mapsto w^{-1}$. That is, $u \leq_R w$ if and only if $u^{-1} \leq_L w^{-1}$.

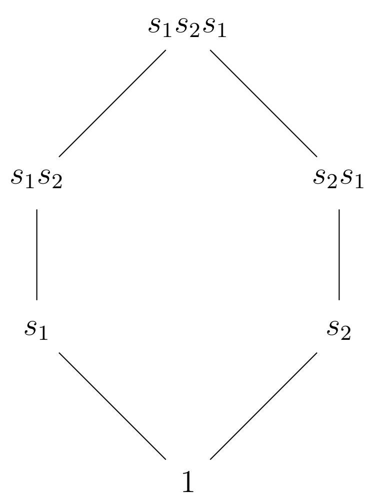


Figure 1. Right weak order on A_2.

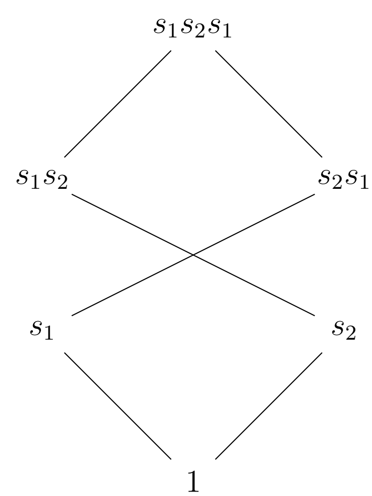


Figure 2. Left weak order on A_2.

Recall that we have a criterion for weak order in terms of inversion sets:

Proposition. We have $u \leq_R w$ if and only if $\mathrm{inv}(u) \subseteq \mathrm{inv}(w)$.

Proof. ($\Rightarrow$) First suppose that $u \leq_R w$. By definition, we have v with $w = uv$ and $l(w) = l(u)+l(v)$. Let $s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced word for v and consider the sequence u, us_{i_1}, $us_{i_1}s_{i_2}$, …, $us_{i_1}\cdots s_{i_k} = uv = w$, which is increasing in length (hence number of inversions) at each step. At each step we have:

$$\mathrm{inv}(us_{i_1}) = \mathrm{inv}(u) \cup \{us_{i_1}u^{-1}\}$$
$$\mathrm{inv}(us_{i_1}s_{i_2}) = \mathrm{inv}(us_{i_1}) \cup \{(us_{i_1})s_{i_2}(us_{i_1})^{-1}\}$$
$$\vdots$$

Since we are only adding inversions in each step, we conclude that $\mathrm{inv}(u) \subseteq \mathrm{inv}(w)$.

($\Leftarrow$) On the other hand, suppose $\mathrm{inv}(u) \subseteq \mathrm{inv}(w)$. Then as usual, choose a geometric representation of W, choose points $x \in D^\circ$ and $y \in wD^\circ$ and consider the line $\overline{xy}$. The line only crosses the (finitely many) hyperplanes that separate D° from wD°, and only crosses them in length-increasing directions. $\square$

We now describe a few conventions for S_n. Firstly, we denote by s_i the transposition $(i\ i+1)$; we will also use 1-line notation to describe elements of S_n. The one line notation $w_1 w_2 \cdots w_n$ means $w(i) = w_i$. Using this notation, right multiplication by s_k interchanges the numbers in **positions** k and $k+1$, while multiplication on the left by s_k interchanges the **values** k and $k+1$.

We can see this in the diagrams below. Moving from 132 to 312 in the right order diagram is multiplication by s_1, and it interchanges 1 and 3, which are the numbers in positions 1 and 2. Moving from 132 to 231 in the right order diagram is multiplication on the left by s_1, and it interchanges 1 and 2, which are the numbers with values 1 and 2.

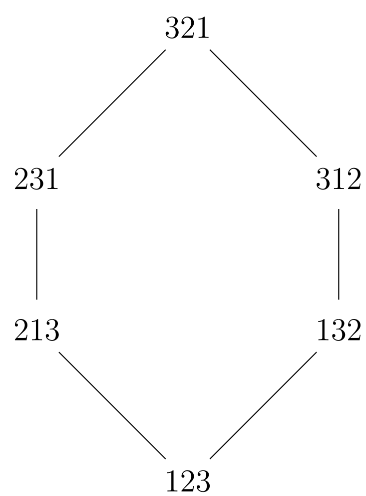


FIGURE 3. Right weak order on S_3, using 1-line notation.

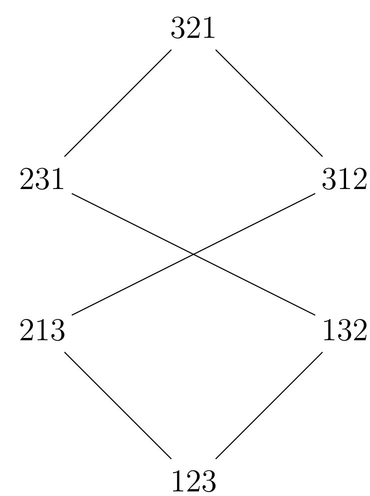


FIGURE 4. Left weak order on S_3, using 1-line notation.

Using 1-line notation, it is easy to relate the diagrams for weak order to the hyperplane arrangement: $w(1)w(2)\cdots w(n)$ corresponds to the region $x_{w(1)} < x_{w(2)} < \cdots < x_{w(n)}$.

A couple more notes about conventions:

(1) The action on the dual space is $w \cdot x_i = x_{w^{-1}(i)}$. We act by the inverse so that the action is a left action.
(2) $(i\ j) \in \mathrm{inv}(w)$ for $i < j$ if i comes after j in the 1-line notation for w.

Recall that we can realize $B_n \subset S_{2n}$ as acting on the set $\{-n, -n+1, \ldots, -1, 1, \ldots, n-1, n\}$ via permutations $w \in S_{2n}$ with the restriction that $w(-i) = -w(i)$. In 1-line notation, w is denoted by $w(-n)\cdots w(-2)w(-1)w(1)w(2)\cdots w(n)$. The inversions of B_n are the same as those of S_{2n}, except we identify $(i\ j)$ and $(-i\ \ -j)$.

Remark. In our diagrams for B_n, we usually omit where the negative numbers map because this is determined by the required symmetry of our maps. In the diagram above, we would omit the first two numbers of each element written in 1-line notation.

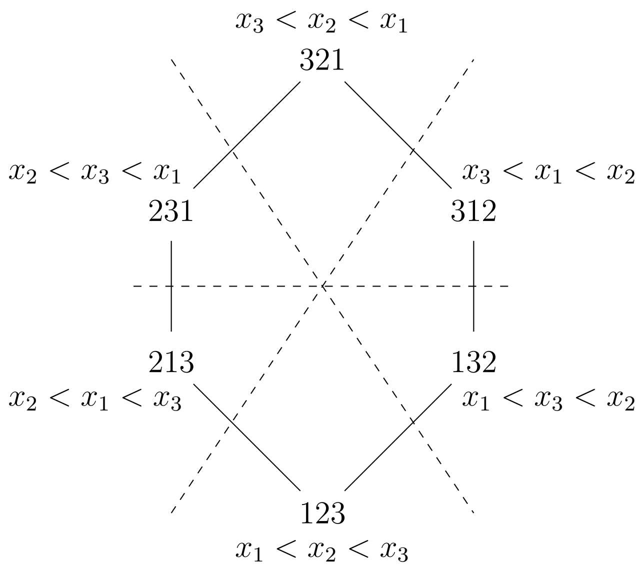


FIGURE 5. The correspondence between the 1-line notation for elements of S_3 and the regions of A_2.

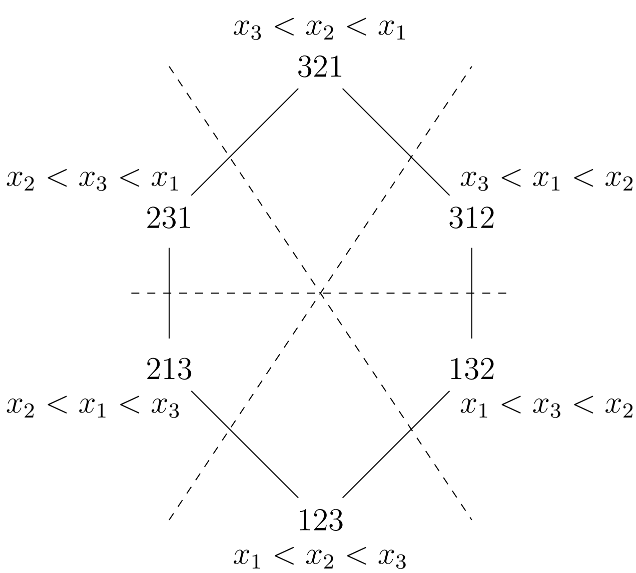


FIGURE 6. The correspondence between the 1-line notation for elements of S_3 and the regions of A_2.

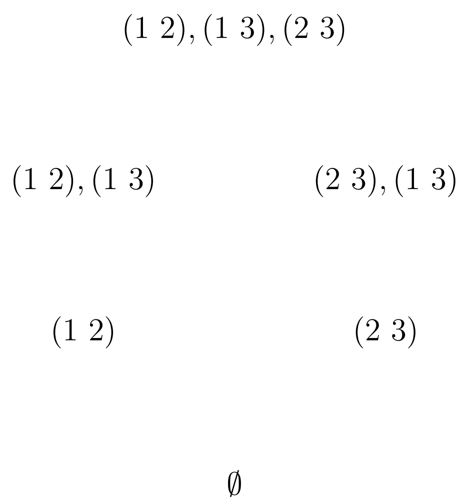


FIGURE 7. The inversions corresponding to each element of S_3, or equivalently, each region of A_2.

Recall that $D_n \subset B_n \subset S_{2n}$, and D_n consists of the permutations of B_n which (as permutation matrices) have an even number of $(-)$ signs. In terms of 1-line notation, this means that there are an even number of $(-)$ signs among $w(1), w(2), \ldots, w(n)$.

The diagram for D_n is the same. We just discard the inversions $(i\bar{i})$, which corresponds to forgetting the hyperplanes $\{x_i = -x_i\} = \{x_i = 0\}$ of B_n.

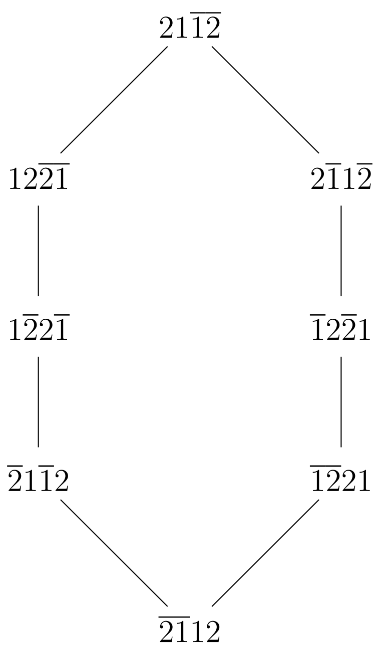


FIGURE 8. Right weak order on B_2, using 1-line notation.

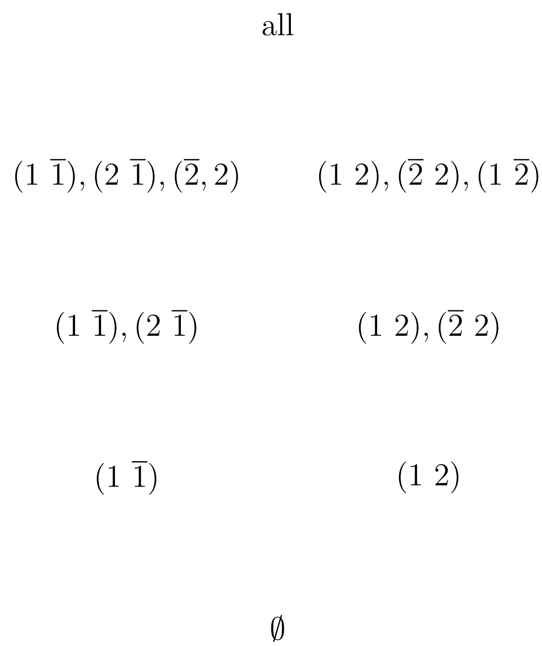


FIGURE 9. The corresponding inversions in B_2.

Now we move on to the affine groups. Recall that $\tilde{A}_{n-1}$ can be thought of as the set of maps $f : \mathbb{Z} \to \mathbb{Z}$ with $f(i+n) = f(i) + n$ and $\sum_{i=1}^{n} f(k) = \sum_{i=1}^{n} k$. To see which inversions are present, we look at what is out of order after applying f. Our restrictions on f force integers congruent mod n to be in order, so it suffices to look at what happens mod n. Moreover, $(i\ j)$ being out of order is equivalent to $(i+n\ j+n)$ being out of order. We'll abbreviate $\cdots(i\ j)(i+n\ j+n)(i+2n\ j+2n)\cdots$ to $(i+kn\ j+kn)$.

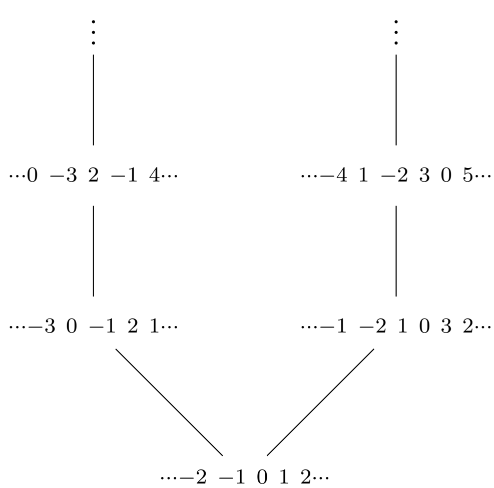


FIGURE 10. The right weak order on $\tilde{A}_1$, in 1-line notation.

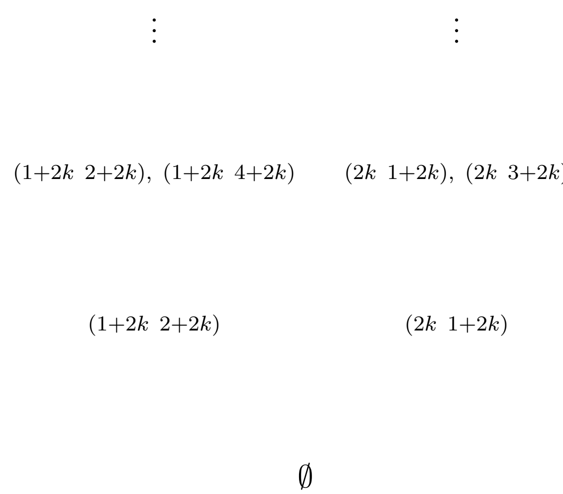


FIGURE 11. The corresponding inversions in $\tilde{A}_1$.

Recall that $\tilde{C}_n \subset \tilde{A}_{2n-1}$ can be thought of as the maps $f : \frac{1}{2} + \mathbb{Z} \to \frac{1}{2} + \mathbb{Z}$ with the restrictions that $f(i + 2n) = f(i) + 2n$ and $f(-i) = -f(i)$. Note that some other sources define $\tilde{C}_n$ as maps $f : \mathbb{Z} \to \mathbb{Z}$ with $f(i + 2n) = f(i) + 2n$ and $f(1 - i) = 1 - f(i)$.

We have the inclusions

$$\tilde{D}_n \subset \tilde{B}_n \subset \tilde{C}_n \subset \tilde{A}_{2n-1}$$

To go from $\tilde{C}_n$ to $\tilde{B}_n$, we forget the inversion $(i\ j)$ with $i + j \equiv 4n\ (8n)$. To go from $\tilde{B}_n$ to $\tilde{D}_n$, we forget the inversions $(i\ j)$ where $i + j \equiv 0\ (4n)$.

Definition. A ***poset*** is a set P with a relation $\leq$ which is

(1) reflexive ($x \leq x$)
(2) anti-symmetric ($x \leq y$ and $y \leq x$ implies $x = y$)
(3) transitive ($x \leq y \leq z$ implies $x \leq z$)

Note that some books may define $<$ instead.

Definition. We say that x ***covers*** y, written $x \gtrdot y$, if $x > y$ and $\nexists z$ with $x > z > y$.

The ***Hasse diagram*** of P is the directed graph with $x \to y$ if $x \gtrdot y$. In the previous diagrams, edges "point down."

Definition. A poset P is a ***lattice*** if any two elements $x, y \in P$ have

(1) a join, written $x \vee y$, which is an element of P that satisfies $x \vee y \geq x, y$ and if $z \geq x, y$, then $z \geq x \vee y$;
(2) a meet, written $x \wedge y$, which is an element of P that satisfies $x \wedge y \leq x, y$ and if $z \leq x, y$, then $z \leq x \wedge y$.

October 21 – Introduction to lattices

Remark. Before proceeding with the main material of today's class, we take a moment to mention the word "permutohedron", which hasn't come up yet, but will be useful as we move towards talking about other polytopes. While the hyperplane arrangement is probably the "correct" object to consider in talking about the geometry of a Coxeter group, polytopes provide a dual option.

We consider our usual setup of a Coxeter group W, a Cartan matrix, roots $\{\alpha_i\}$, and coroots $\{\alpha_i^\vee\}$, and define the chamber D° as before. We assume that the roots, and likewise the coroots, are linearly independent.

Now let $\rho \in D^\circ$, and define P to be the convex hull of the orbit $W\rho$. We say that P is the ***W-permutohedron*** (though perhaps it should be *a* W-permutohedron). The examples of A_2 and A_3 are pictured in Figures 12 and 13. We remark that the edge length of the A_2 permutahedron are $b - a$ and $c - b$; Professor Speyer recommends NOT drawing them to be the same length.

The W-permutohedron is dual to the hyperplane arrangement, in the following sense. If $P \subset V$ is any polytope, and F is a face, then the ***normal cone*** of F is

$$N_F(P) := \{\theta \in V^\vee \mid \langle \theta, \cdot \rangle \text{, restricted to } P\text{, attains its maximum on } F\}$$

In our case, the k-dimensional faces of the W-permutohedron are dual to the $(n - k)$-dimensional cones of the hyperplane arrangement.

Now we return to the subject of lattices. Our goal today is to show that the weak order on a finite Coxeter group is no mere poset, but in fact a lattice (and even outside finite type, it is a semilattice, which we'll introduce later).

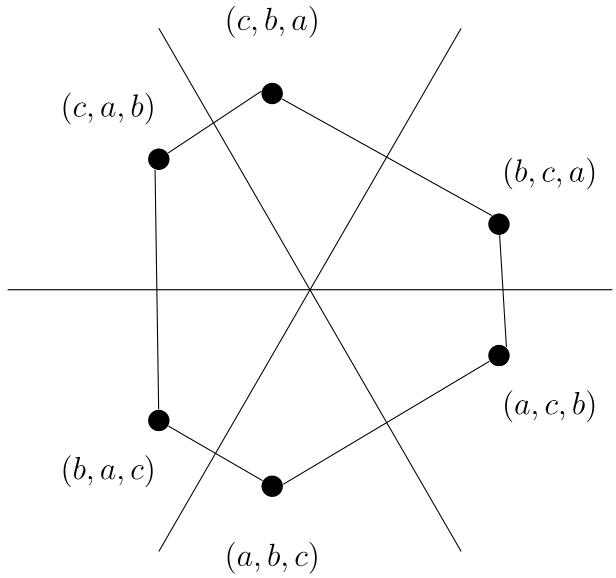


FIGURE 12. An A_2-permutohedron.

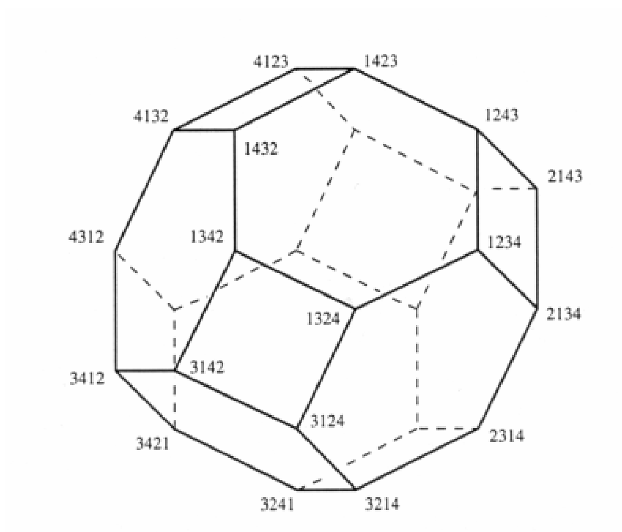


FIGURE 13. An A_3-permutohedron.

Definition. A poset P is a ***lattice*** if for any elements x, y, there exist elements $x \vee y$ (the ***join***) and $x \wedge y$ (the ***meet***) such that:

- $x \vee y \geq x$ and y, and if $z \geq x$ and y, then $z \geq x \vee y$ (i.e., $x \vee y$ is the least upper bound of x and y)
- $x \wedge y \leq x$ and y, and if $z \leq x$ and y, then $z \leq x \wedge y$ (i.e., $x \wedge y$ is the greatest lower bound of x and y)

To remember which of $\vee$ and $\wedge$ is which, one can think of $\cup$ and $\cap$ which are, respectively, the join and meet on the lattice of subsets of a set.

Clearly the poset structure determines the join and meet operations. In fact, the reverse is also true: the poset structure in a lattice is completely determined by the join and meet operations, because

$$x \leq y \Leftrightarrow x = x \wedge y \Leftrightarrow y = x \vee y$$

We'll examine in a homework problem what conditions on binary operations $\vee$ and $\wedge$ result in a valid poset defined this way.

Since we like dealing with infinite Coxeter groups, we need to discuss meets and joins of infinite sets as well. For any subset X of a poset P, we'll write $\bigvee X$ for an element z of P such that $z \geq x$ for all $x \in X$ and such that, if $y \geq x$ for all $x \in X$, then $z \geq y$. We define $\bigwedge X$ dually.

Definition. A ***complete lattice*** is a poset in which arbitrary subsets have joins and meets.

This is certainly true of a finite lattice[1]. The only nonobvious aspect of the definition we have to check is what the join and meet of the empty set are. In fact, for a finite lattice L,

$$\bigwedge \emptyset = \bigvee L, \qquad \bigvee \emptyset = \bigwedge L$$

The former element is the unique maximal element of the lattice, and we call it $\widehat{1}$. The latter is the unique minimal element, and we call it $\widehat{0}$.

In order to show that a poset is a lattice, it's enough to show that it has meets. Precisely, we have

Lemma. If a poset P has a unique maximal element $\widehat{1}$, and the meet of any subset exists[2], P is a complete lattice.

Proof. (sketch) We can define the join of a subset X by

$$\bigvee X = \bigwedge \{y \mid y \geq x \ \forall x \in X\}.$$

In other words, the least upper bound of X is the greatest lower bound of all upper bounds of X. □

Remark. One neat example of this lemma is given by the lattice of subgroups of a group. We know that this poset has all meets, as the intersection of subgroups is a group; from there, we define the join of a set of subgroups to be the intersection of all subgroups containing them, which is exactly the construction used in the lemma.

Still, this lemma is not quite enough for showing that weak order is a lattice, because it's not immediately clear how to construct joints *or* meets. So we turn to a slightly stronger lemma which is both fun and cute.

Lemma (Björner, Edelman, Ziegler)**.** Let P be a finite poset with a unique maximal element $\widehat{1}$ such that, if $x \lessdot z$ and $y \lessdot z$, $x \wedge y$ exists. Then P is a lattice.

Proof. It suffices to show that $x \wedge y$ exists for any $x, y \in P$. Then since P is a finite poset, we know all meets exist, and we can use the previous lemma.

Now we want to prove this by induction; thus, in order to do induction on our poset, we complete it to a total order in some way. So we label the elements of the poset $x_1, \ldots, x_N$ such that if $x_i \leq x_j$, then $i \leq j$. We'll prove that if $y, z \leq x_k$, then $y \wedge z$ exists, by induction

[1] Except for the empty poset, but let's not worry about that too much.

[2] This hypothesis is a little redundant, because $\widehat{1}$ is the meet of the empty set.

on k. Here we note that, in order for this to actually prove the desired result, we need a unique maximal element to exist, so that the statement for $k = N$ proves the result.

In the base case $k = 1$, if $y, z \leq x_1$, we actually have $y = z = x_1$, and $y \wedge z = x_1$.

Now we proceed to the induction step. Given $y, z \leq x_k$, we choose elements x_i and x_j such that $y \leq x_i \lessdot x_k$ and $z \leq x_j \lessdot x_k$. By the hypothesis of the lemma, $x_i \wedge x_j$ exists. Now $y, x_i \wedge x_j \leq x_i$, so by the induction hypothesis, $y \wedge (x_i \wedge x_j)$ exists. Likewise, $z, x_i \wedge x_j \leq x_j$, so by the induction hypothesis $z \wedge (x_i \wedge x_j)$ exists. Finally, we can apply the induction hypothesis once more and conclude that $(y \wedge (x_i \wedge x_j)) \wedge (z \wedge (x_i \wedge x_j)) = y \wedge (x_i \wedge x_j) \wedge z$ exists. This sequence of reasoning is captured by the following diagram, where straight lines represent cover relations and wavy lines represent chains.

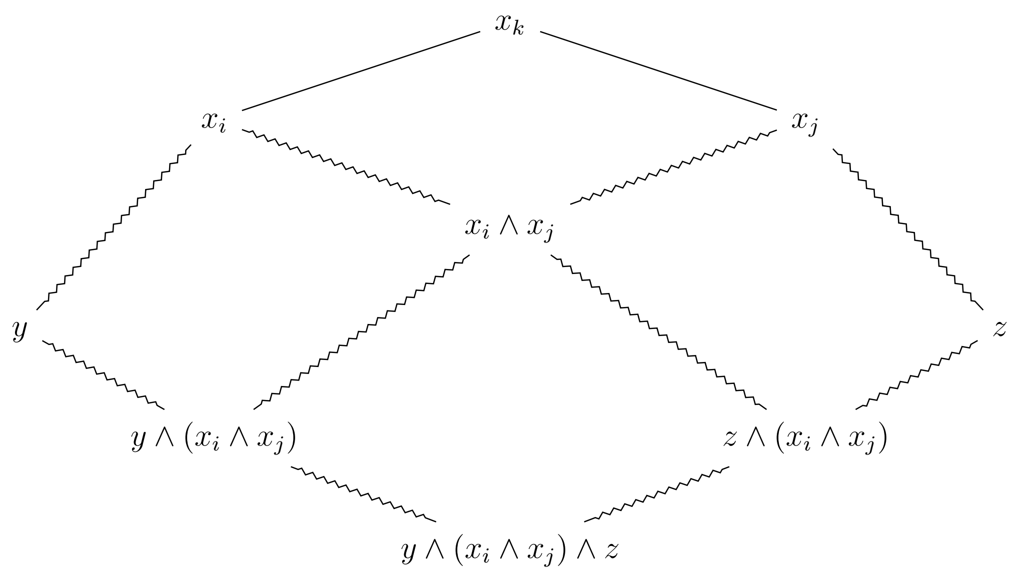


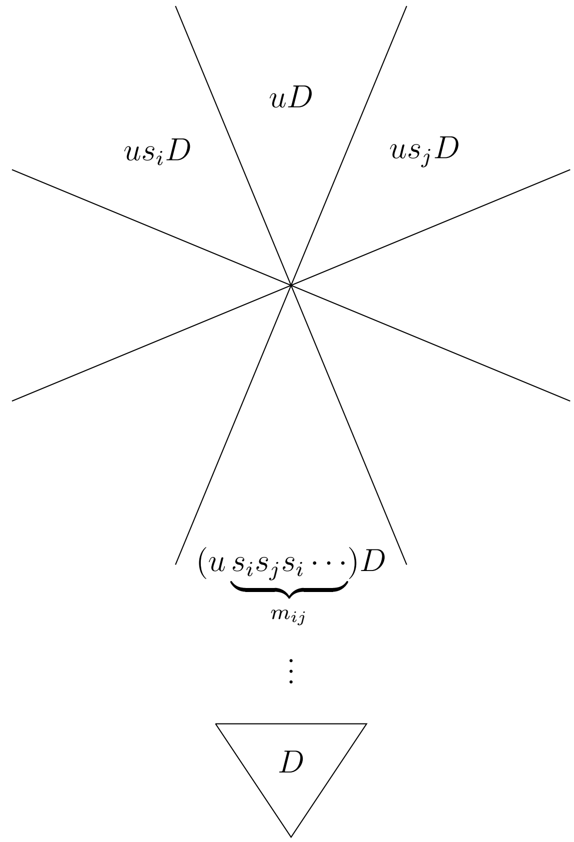
FIGURE 14. The structure of the proof of the BEZ lemma. We show the displayed meets exist from top to bottom.

But now we claim that $y \wedge (x_i \wedge x_j) \wedge z = y \wedge z$. Indeed, any lower bound of y and z is also a lower bound of x_i and x_j, and thus also $x_i \wedge x_j$, so wedging with $x_i \wedge x_j$ is redundant. ☐

Remark. The structure shown in Figure 22 shows up in multiple other arguments: the proof of the Jordan-Hölder theorem for subgroups, the Diamond Lemma in ring theory, and Buchberger's S-pair reducing criterion for Gröbner bases. So this diagram is worth remembering.

So what does this lemma allow us to do in the case of a Coxeter group? In the case of W finite, we can very directly see that weak order makes it a lattice. Checking the conditions of the BEZ lemma:

- We have a unique maximal element $\widehat{1}$, namely the longest word w_0.
- Now suppose we have two cover relations $u \gtrdot_R us_i$, $u \gtrdot -Rus_j$ (so that $\ell(us_i) = \ell(us_j) = \ell(u) - 1$). We want to show that the meet $us_i \wedge us_j$ exists. We claim that it is given by $u \underbrace{s_i s_j s_i \cdots}_{m_{ij}}$, as illustrated in the diagram below.



Certainly $u \underbrace{s_i s_j s_i \cdots}_{m_{ij}}$ is a lower bound of us_i and us_j in the weak order. To see that it is the greatest lower bound, suppose that v is any other lower bound of us_i and us_j. Then we have $\mathrm{inv}(v) \subset \mathrm{inv}(us_i) \cap \mathrm{inv}(us_j)$. Then $\mathrm{inv}(v)$ contains neither us_iu^{-1} (since this is not an inversion of us_i) nor us_ju^{-1} (since this is not an inversion of us_j). Thus vD lies on the same side of the associated fixed hyperplanes as $u \underbrace{s_i s_j s_i \cdots}_{m_{ij}}$. Thus $v \leq_R u \underbrace{s_i s_j s_i \cdots}_{m_{ij}}$.

Corollary. For W *any* Coxeter group and element $u, v \in W$ with $u \leq_R v$, the interval

$$[u, v] := \{x \in W \mid u \leq_R x \leq_R v\}$$

is a lattice.

Proof. Note first that $[u, v]$ is finite: any element must have an inversion set contained in that of v, which is a finite set. Thus we can apply the BEZ lemma. We can use the same argument as above: we only need to know that the construction in our argument doesn't leave the interval $[u, v]$. But if $ws_i \lessdot_R w$, $ws_j \lessdot_R w$ are elements as in the above lemma in $[u, v]$, u is a lower bound for both of them, and thus $ws_i \wedge ws_j \geq_R u$. □

Corollary. Let W be a Coxeter group, and $X \subset W$ any subset. Then:

1) if $X \neq \emptyset$, $\bigwedge X$ exists.
2) if X has an upper bound, $\bigvee X$ exists.

Proof. To prove 1), suppose $x_0 \in X$. Then we apply the above corollary to the interval $[e, x_0]$, and get

$$\bigwedge X = \bigvee \{y \mid \forall x \in X, y \leq_R x\}$$

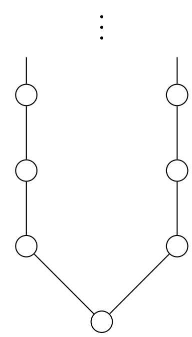
That is, once we have some element in X, all lower bounds for X are forced into the interval below that element, and we can take their join there.

Now to prove 2), Let z be an upper bound for X. Then the set of all upper bounds is nonempty, and so by part 1) we can take their meet and obtain

$$\bigvee X = \bigwedge \{y \mid \forall x \in X, x \leq_R y\}.$$

□

A poset with the two properties articulated in the previous corollary is called a ***complete semilattice*** – a poset in which nonempty meets exist, and joins exist as long as some upper bound does. For example, the weak order on the infinite dihedral group looks like this:



Two elements from the two different branches don't have a join, but that's because they don't have *any* common upper bound.

October 23 – Tamari Lattices and Related Topics

Today's class began with an introduction to Tamari Lattices, denoted T_n, with an overview of their history and discovery. We then described the Loday Construction, which defines a map from the symmetric group $S_n \xrightarrow{\pi} T_n$. Since this lecture is historical, I'll try to do a better job than usual citing sources. To begin with, let me recommend Nathan Reading's *From the Tamari lattice to Cambrian lattices and beyond*[3] and Ceballos, Santos and Ziegler, *Many non-equivalent realizations of the associahedron*[4], two sources which I found very helpful in preparing this lecture.

Definition. Consider all the ways of parenthesizing a product of $n + 1$ terms. Let these parenthesizations be vertices of a graph, with an edge between two parenthesizations signifying that they differ by a single association. This graph is called a ***Tamari Lattice***, and is denoted T_n.

Tamari Lattices are in fact directed graphs, given that we choose an orientation of the edge $x(yz) \longrightarrow\!\!\!\!\!- (xy)z$. We choose the orientation as follows: $x(yz) \longrightarrow (xy)z$.

[3]published in "Associahedra, Tamari lattices and related structures", 293–322, Prog. Math. Phys., 299, Birkhäuser/Springer, Basel, 2012

[4]*Combinatorica* **35** (2015), no. 5, 513–551

Example. In T_2, the vertices correspond to each way to parenthesize a product of 3 terms. We have only two vertices, with an edge connecting them:

$$a(bc) \longrightarrow (ab)c$$

Example. In T_3, there exists 5 parenthesizations of a product of 4 terms.

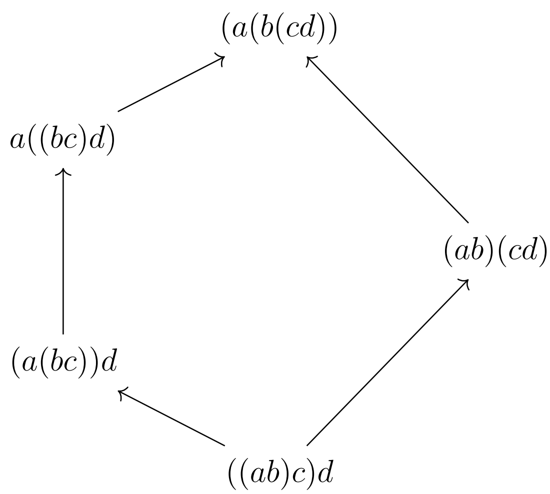


FIGURE 15. The Tamari Lattice T_3.

It is not immediately clear that a Tamari Lattice is in fact a lattice. It is easy to verify that T_n is acyclic as a directional graph, so taking the transitive closure of T_n turns a Tamari Lattice into a poset.

Theorem. (Tamari, 1951[5]) T_n is a lattice.

The order of T_n can be counted via the Catalan numbers. The Catalan numbers are prolific within the realm of combinatorics, and count a myriad of different things. Today, we focus on 3 of them, one of which being the order of T_n. The others are the number of Binary Trees with $n+1$ leaves, and the number of triangulations of an $(n+2)$-gon. Later, a belief formed that a Tamari Lattice should be realized as an ***Associahedron***.

Definition. An Associahedron is an $n-1$ polytope whose vertices are indexed by parenthesizations of a product of $n-1$ terms, where edges correspond to differing by a single association (as with Tamari Lattices). In addition, $n-k$ faces should correspond to parenthesizations with precisely k parenthesis pairs.

The associahedra corresponding to T_2, T_3 are identical to the above. The associahedron corresponding to T_4 was detailed in the handout given in class. Associahedra have a rich history and are connected to Coxeter Groups and Cluster Algebras. A brief history is detailed below:

- (Stasheff, 1963[6]) First constructed an associahedron as a purely topological object.

[5]Ph. D. thesis, later published in 1962 as "The algebra of bracketings and their enumeration", Nieuw Archief voor Wiskunde, Ser. 3, 10: 131–146

[6]"Homotopy associativity of H-spaces. I, II", Transactions of the American Mathematical Society, 108: 293–312 (1963)

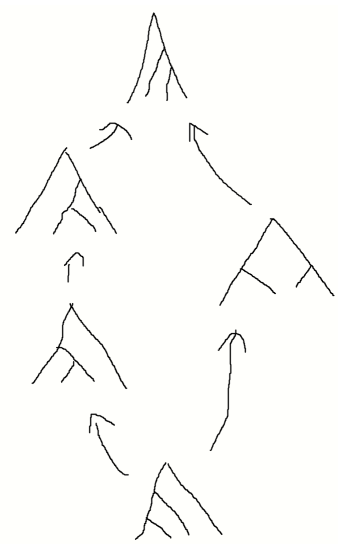


FIGURE 16. Binary Trees with 4 Leaves.

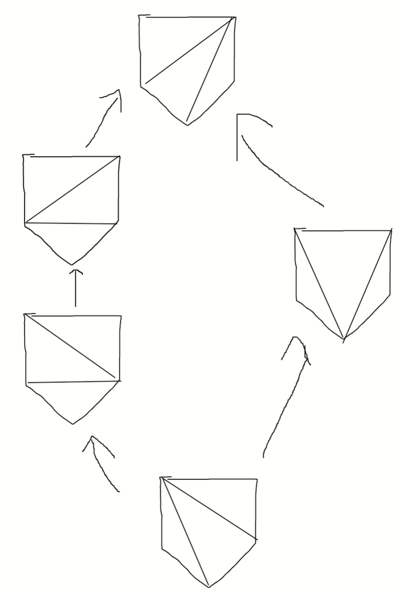


FIGURE 17. Triangulations of a 5-gon.

- (Haiman, 1984, unpublished) and (Lee, 1989[7]) constructed the associahedron as a polytope.
- (Gelfand-Kaparanov-Zelensky, 1990[8]) and (Billera-Sturmfels, 1992[9]) associated the triangulations of any points in $\mathbb{R}^d$ with a polytope, denoted the secondary polytope.

[7] "The associahedron and triangulations of the n-gon", European J. Combinatorics 10 (1989),no. 6, 551-560.

[8] "Newton polytopes of principal A-determinants", Soviet Math. Doklady 40 (1990), 278–281. See also their book "Discriminants, Resultants, andMultidimensional Determinants", 1994

[9] Fiber polytopes.Annals of Math. 135 (1992), 527–549

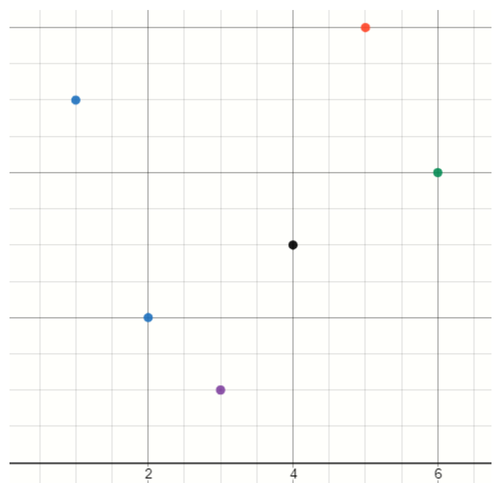
- Motivated by Cluster Algebras, a notion of W-associahedron for any finite crystallographic Coxeter group is introduced in (Chapaton-Fomin-Zelevinsky, 2002[10]).

We now develop the ***Loday Construction***, which constructs a projection map $S_n \xrightarrow{\pi} T_n$, with an associated map from a permutahedron (resp. S_n) to an associaherdron (resp. T_n). Such a construction was introduced by (Tonks, 1997[11]), and written and detailed very clearly by (Loday, 2004[12]).

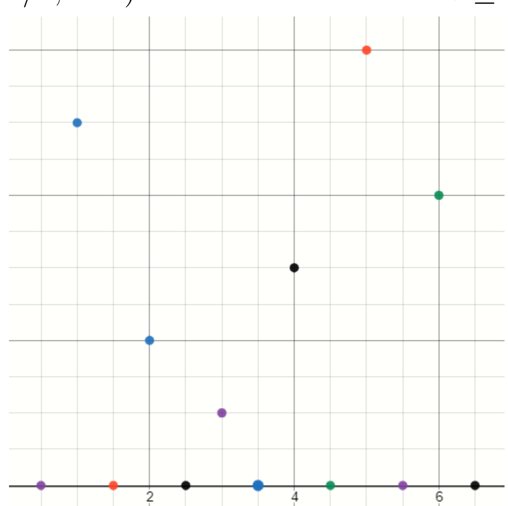
Given $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we can build a binary tree with $n+1$ leaves, defined in $\mathbb{R}^2$. Internal vertices of the tree are of the form (i, x_i), with leaves at $(1/2, -\infty), (3/2, -\infty), \ldots, (n+1/2, -\infty)$. To define the construction, we first consider the case where $x_i \neq x_j$ where $i \neq j$. If this is the case, we can totally order the x_i. We consider the case in $\mathbb{R}^6$ explicitly. Suppose that

$$x_3 < x_2 < x_4 < x_6 < x_1 < x_5$$

We graph (i, x_i) first:



We identify points on $(n+1/2, -\infty)$ at the x-axis for each $0 \leq n \leq 6$:

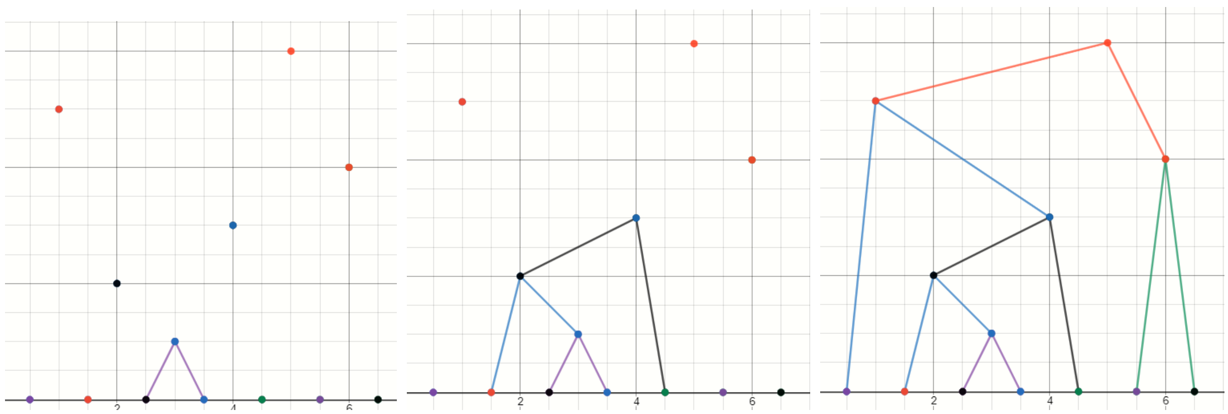


[10]"Polytopal realizations of generalized associahedra", *Canad. Math. Bull.* **45** (2002), no. 4, 537–566

[11]Relating the associahedron and the permutohedron, in *Operads: Proceedings of Renaissance Conferences* (Hartford, CT/Luminy, 1995), J.-L. Loday, J.D. Stasheff and A.A.Voronov, eds., Contemp. Math., vol. 202, Amer. Math. Soc., Providence, RI, 1997, 33–36

[12]"Realization of the Stasheff polytope", *Arch. Math.* (Basel) **83** (2004), no. 3, 267–278

Then, beginning at the smallest x_i, (in this case, x_3), we draw an edge to its closest neighbors in the x-coordinate, then continue to the next x_i.



Clearly, the tree associated to $(x_1, \ldots, x_n)$ only depends on which region of the S_n hyperplane arrangement $(x_1, \ldots, x_n)$ lies in. We depict the result for S_3 in Figure 26.

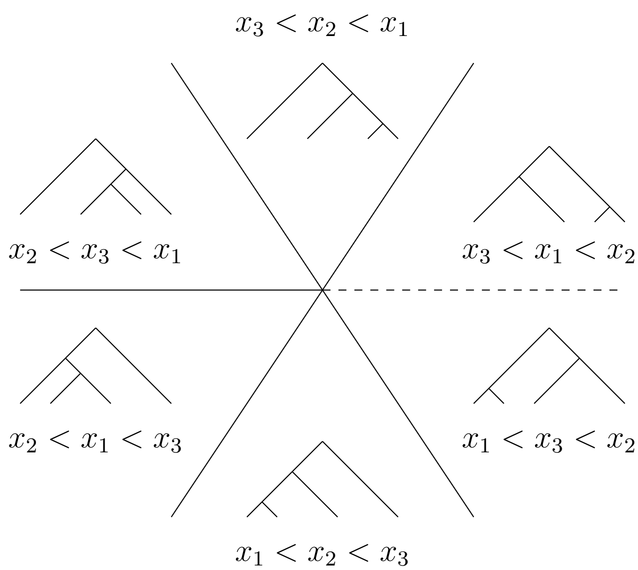
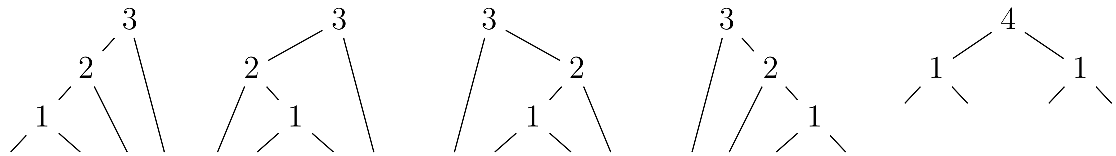


FIGURE 18. Loday's map from $\mathbb{R}^3$ to T_3 as a coarsening of the S_3-hyperplane arrangement

We note that the cones $x_1 < x_3 < x_2$ and $x_3 < x_1 < x_2$ give rise to trees which are topologically the same, differing only in what height we draw their vertices at. In general, Loday's map gives us a map $\pi : S_n \to T_n$ and thus a coarsening of the S_n hyperplane arrangement.

Loday also uses this construction to give coordinates for the vertices of the associahedron. Number the internal vertices of our tree T as $(v_1, v_2, \ldots, v_n)$ from left to right. Let $c_k(T)$ be the number of pairs of leaves (i, j) where i is a left descendent of v_k and j is a right descendent.



So our vertices are $(1, 2, 3)$, $(2, 1, 3)$, $(3, 1, 2)$, $(3, 2, 1)$ and $(1, 4, 1)$. The dual fan is the corresponding coarsening of the S_n hyperplane arrangement. See Figure 19.

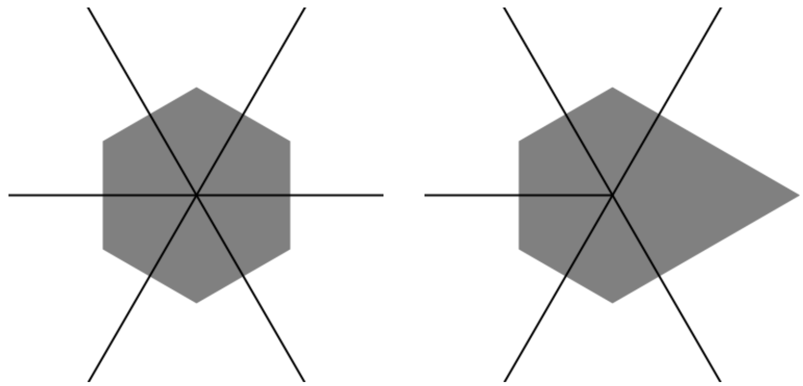


FIGURE 19. The S_3 permutahedron and associahedron, with their normal fans

Since S_n and T_n are both lattices, it makes sense to ask whether π is a lattice homomorphism. It is, as proved by Reading in 2004.

Theorem. (Reading, 2004) π is a lattice homomorphism, which means that

$$\pi(x \vee y) = \pi(x) \vee \pi(y)$$

and

$$\pi(x \wedge y) = \pi(x) \wedge \pi(y)$$

for all x, y.

This procedure of relating polytopes and lattices is one of many.

(1) The Cyclohedron (is a polytope whose vertices correspond to triangulations of a $2n$-gon with symmetry under $180°$ rotation). This was first devised by Taubes, then with the help of Reiner, was made into a polytope, then a lattice. Moreover, the cyclohedron graph turned out to be orientable in many ways, all of which gave lattices.
(2) For Gelfand-Kaparanov-Zelensky, coordinates of the associahedron polytope depend on how the $n+2$ points lie in $\mathbb{R}^2$. Different $n+2$-gons give different lattice structures.
(3) Fomin-Zelevinsky associate a Cluster Algebra to any finite crystallographic Cartain Matrix, and thus a polytope. A natural orientation of this polytope follows from the chosen orientation of the Coxeter diagram.
(4) Given a finite coxeter group W, an orientation Ω of Γ, Reading defines a quotient

$$W \xrightarrow{\pi} \mathrm{Camb}(\Omega)$$

Where $\mathrm{Camb}(\Omega)$ is a ***Cambrian Lattice***. Later, Holweg, Lange, and H. Thomas defined a corresponding associahedron. Reading and Speyer later generalized this result, removing the hypothesis that $|W| < \infty$, and connects these to Cluster Algebras.

October 25 – Lattice Congruences

Definition. Let L_1 and L_2 be lattices. A ***lattice homomorphism*** is a map $\sigma : L_1 \to L_2$ such that $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$ and $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$. If L_1 and L_2 are complete lattices, a ***complete lattice homomorphism*** has $\sigma(\bigvee X) = \bigvee \sigma(x)$ and $\sigma(\bigwedge X) = \bigwedge \sigma(X)$ for all $X \subset L_1$.

Note that since meet and join determine the partial order, (complete) lattice homomorphisms are order-preserving. We also have that $\sigma(\hat{0}) = \hat{0}$ and $\sigma(\hat{1}) = \hat{1}$.

Since the image of σ is a sublattice of L_2, we can factor σ as a surjection followed by an injection. We focus on surjective lattice homomorphisms. Given $\sigma : L_1 \to L_2$, we get an equivalence relation θ (or $\equiv_\theta$) on L_1 where $x \equiv_\theta y$ if $\sigma(x) = \sigma(y)$. As a set, we can identify $\sigma(L_1)$ with L_1/θ. Knowing L_1 and θ determines $\vee$ and $\wedge$ as functions on L_1/θ, so (L_1, θ) determines the partial order on L_1/θ. We'll describe quotient lattices via describing all such equivalence relations.

So: what equivalence relations θ on a lattice L give a lattice structure on L/θ? A rephrasing of the condition is the following definition.

Definition. An equivalence relation θ on a lattice L is a ***lattice congruence*** if whenever $x_1 \equiv_\theta y_1$ and $x_2 \equiv_\theta y_2$, then $x_1 \vee x_2 \equiv_\theta y_1 \vee y_2$ and $x_1 \wedge x_2 \equiv_\theta y_1 \wedge y_2$.

Definition. An equivalence relation θ on a complete lattice L is a ***complete lattice congruence*** if for all index sets I and maps $i \mapsto x_i$, $i \mapsto y_i$ from $I \to L$ such that $x_i \equiv_\theta y_i$ for all i, then we have

$$\bigwedge_{i\in I} x_i \equiv_\theta \bigwedge_{i\in I} y_i \quad \text{and} \quad \bigvee_{i\in I} x_i \equiv_\theta \bigvee_{i\in I} y_i.$$

One might be worried that lattice congruences for infinite lattices could be pretty wild. But the next lemma shows that things aren't as bad as they could be.

Lemma. Let L be a complete lattice, θ a complete lattice congruence, and X an equivalence class of θ. Then

$$X = \left[\bigwedge X, \bigvee X\right] = \left\{z \in L \mid \bigwedge X \leq z \leq \bigvee X\right\}.$$

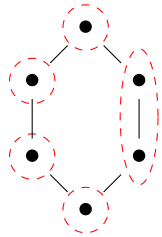
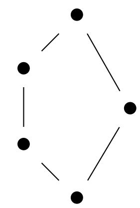
Proof. If $z \in X$, then by definition $\bigwedge X \leq z \leq \bigvee X$. Conversely, note that since θ is a complete lattice congruence, $\bigwedge X \equiv_\theta \bigvee X \in X$, since for any $y \in X$, if we index X by I, then we have maps $i \mapsto x_i$, $i \mapsto y$, and so $\bigwedge_{i\in I} x_i \equiv_\theta \bigwedge_{i\in I} y = y$ and $\bigvee_{i\in I} x_i \equiv_\theta \bigvee_{i\in I} y = y$. Thus if $z \in [\bigwedge X, \bigvee X]$, then $z \wedge \bigwedge X = \bigwedge X$ and so $z \geq_{L/\theta} \bigwedge X$, and similarly $z \leq_{L/\theta} \bigvee X$. Thus $\bigvee X \equiv_\theta \bigwedge X \equiv_\theta z$. □

This leads to the following sometimes-used notation.

Definition. For $x \in L$, $\pi_\downarrow^\theta(x)$ is the bottom element of the equivalence class of x, and $\pi^\theta_\uparrow(x)$ is the top element.

How can we understand congruences better? Let $|L| < \infty$ for the rest of class, since finite lattices are hard enough. Then a congruence θ is determined by the set of covers $x \lessdot y$ in the Hasse diagram for which $x \equiv_\theta y$. This is because if we have any $z \equiv_\theta w$, then we also have $z \equiv_\theta z \vee w \equiv_\theta w$, and have a finite path of covers going up from z to $z \vee w$, then back down to w (and all of these covers are equivalent in θ by the lemma).

Example. With S_3 mapping to the Tamari lattice, we have that the two nodes corresponding to 312 and 132 are combined in an equivalence class, and every other equivalence class is a singleton.

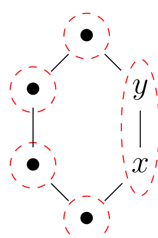
Let $\mathrm{Covers}(L) := \{(x, y) \mid x \lessdot y\}$. Day[13] characterized for which $E \subset \mathrm{Covers}(L)$ there is a lattice congruence collapsing exactly E.

Theorem (Day)**.** $E \subset \mathrm{Covers}(L)$ is the set of collapsed edges of a congruence if and only if the following two conditions hold.

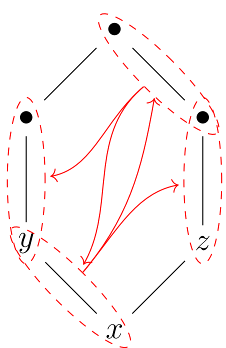
(1) If $(x, y) \in E$, $z \in L$, and $x \vee z \leq x' \lessdot y' \leq y \vee z$, then $(x', y') \in E$.
(2) If $(x, y) \in E$, $z \in L$, and $x \wedge z \leq x' \lessdot y' \leq y \wedge z$, then $(x', y') \in E$.

These conditions are definitely necessary by the lemma from earlier. Before we get into the proof, let's look at some examples.

Example. If we try $E = \{(132,\ 312)\}$ in $\mathrm{Covers}(S_3)$, then for any $z \in S_3$, either $x \vee z = y \vee z \in \{312, 321\}$, or $x \vee z = x$ and $y \vee z = y$. A similar statement follows for meets, and so Day's conditions don't give any new covers and this is the collapsed set of edges for a congruence.



Example. If we try $E = \{(x = 123, y = 213)\}$, then choosing $z = 132$ gives $z \vee y = 321$ but $z \vee x = z$, so this forces two new covers $(132, 312), (312, 321) \in E$. A similar argument shows that this now implies $(213, 231) \in E$. Thus the minimal lattice congruence we get has only two equivalence classes.



[13] Alan Day, Characterizations of finite lattices that are bounded-homomorphic images of sublattices of free lattices. Canadian J. Math. 31 (1979), no. 1, 69–78, Lemma 3.2.

Some side comments: for any lattice L, the set of lattice congruences is a poset via $\theta_1 \leq \theta_2$ if $x \equiv_{\theta_1} y$ implies $x \equiv_{\theta_2} y$. In fact, this is also a lattice! And even better, it's distributive, i.e. $\theta_1 \vee (\theta_2 \wedge \theta_3) = (\theta_1 \vee \theta_2) \wedge (\theta_1 \vee \theta_3)$ and $(\theta_1 \vee \theta_2) \wedge \theta_3 = (\theta_1 \wedge \theta_3) \vee (\theta_2 \wedge \theta_3)$.

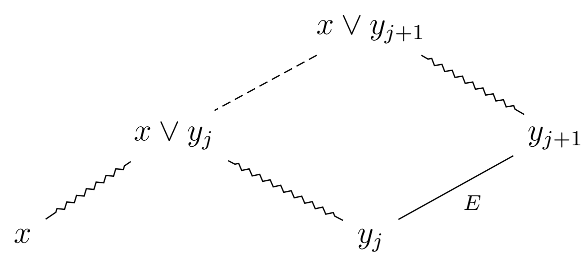
Now onto the proof! We'll write $\implies$ for the transitive closure of Day's conditions on Covers(L).

Proof. Let $E \subset$ Covers(L) such that $(x, y) \in E$ and $(x, y) \implies (x', y')$ means $(x', y') \in E$ (i.e. E satisfies Day's conditions). We'll show there exists a lattice congruence θ such that $E = \{(x, y) \in \text{Covers}(L) \mid x \equiv_\theta y\}$. Try θ the equivalence relation generated by covers in E, i.e. θ generated by $x \equiv y$ for $(x, y) \in E$. We want to show

- that θ is a lattice congruence; and
- that θ doesn't collapse extra edges, i.e. $(x, y) \notin E$ means $x \not\equiv_\theta y$.

First we prove the first bullet. Let $x \equiv_\theta x'$ and $y \equiv_\theta y'$. We want to show that $x \vee y \equiv_\theta x' \vee y'$, and that $x \wedge y \equiv_\theta x' \wedge y'$.

Let $x = x_0 - x_1 - \ldots - x_m = x'$, where (x_i, x_{i+1}) or $(x_{i+1}, x_i) \in E$ for each i. Similarly, let $y = y_0 - y_1 - \ldots - y_n = y'$, where (y_j, y_{j+1}) or (y_{j+1}, y_j) in E for each j. We'll show by induction on $i + j$ that $x_0 \vee y_0 \equiv x_i \vee y_j$. Really we just need to show that $x_i \vee y_j \equiv x_i \vee y_{j+1}$, and WLOG assume $y_j \lessdot y_{j+1}$. Choose a chain $x \vee y_j$ to $x \vee y_{j+1}$. Day's condition 1 gives that every edge in the chain is in E, so we're done. The picture to have in mind is below: the squiggly lines come from paths of edges in E, and of course $(y_j, y_{j+1}) \in E$. Day's condition gives us that the dotted line is a path in E



Next we prove the second bullet. Suppose we have $x \lessdot x'$, and an E-chain $x = x_0 - x_1 - \ldots - x_n = x'$. By joining everything in the chain with x, we can get a new collection of monotone paths in E connecting each $x_i \vee x$ with $x_{i+1} \vee x$,

$$x = x_0 \vee x \rightsquigarrow x_1 \vee x \rightsquigarrow \cdots \rightsquigarrow x_n \vee x.$$

Since each $x_i \vee x \geq x$, the E chain is entirely $\geq x$. Repeat this trick to each entry of the chain by meeting with x' to get a chain where every element is $\geq x$ and $\leq x'$. So every link of the chain is x or x', and in particular $(x, x') \in E$. $\square$

October 28 – Join Irreducibles

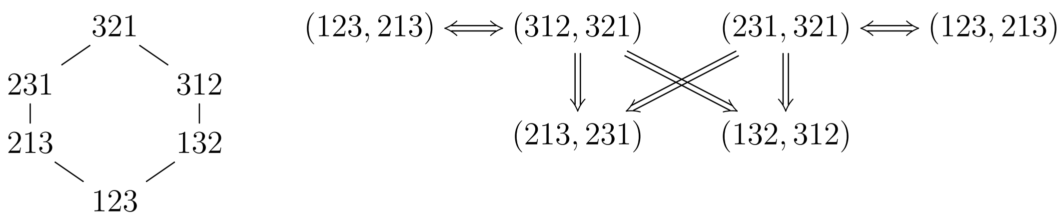
Recall the theorem of Day from last time:

Theorem (Day)**.** $E \subset$ Covers(L) is the set of collapsed edges of a congruence if and only if the following two conditions hold.

(1) If $(x, y) \in E$, $z \in L$, and $x \vee z \leq x' \lessdot y' \leq y \vee z$, then $(x', y') \in E$.
(2) If $(x, y) \in E$, $z \in L$, and $x \wedge z \leq x' \lessdot y' \leq y \wedge z$, then $(x', y') \in E$.

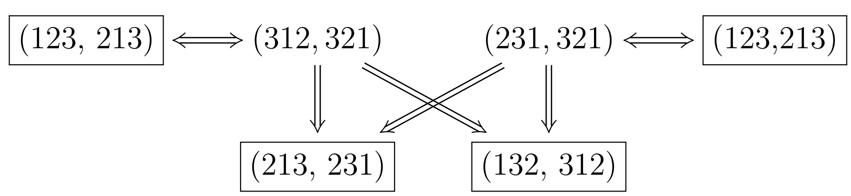
One way we can form such a set $E \subset \text{Covers}(L)$ is by a cover $x \lessdot y$, and seeing what other covers must be in E because of this (in other words, we take the smallest set E that contains (x, y) and satisfies the conditions of the theorem). If we have $x \vee z \leq x' \lessdot y' \leq y \vee z$ for some $z, x', y' \in L$, then we must also have $(x', y') \in E$. We say that the cover (x, y) forces (x', y'). We write $(x, y) \implies (x', y')$ if (x, y) forces (x', y').

Example. Here is the complete list of forcing relations for S_3 (together with the Hasse diagram of S_3, as a reminder).



Note that the relation of forcing gives a preorder (reflexive and transitive relation) on the set of covers. The preorder is not antisymmetric (which can be seen in the above diagram), so it is not a partial order. But it is an easy exercise to prove that a preorder $\preceq$ induces a partial order on the set of equivalence classes defined by $a \sim b \iff a \preceq b$ and $b \preceq a$.

Now, we see that we have reduced the problem of finding lattice congruence relations to finding downward closed sets in this partial order on the equivalence classes. We want to take these equivalence classes and be able to pick out a canonical cover, and to do that, we introduce the notion of a complete join irreducible element. Each join irreducible element will give us a covering pair. Here is the previous example of the forcing preorder of S_3, with the join irreducible covers indicated:



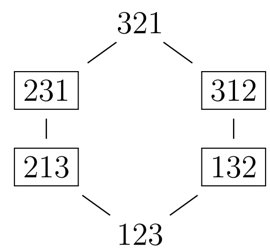
Lemma. Let L be a complete lattice. For $j \in L$, the following are equivalent

(1) $j = \bigvee X \implies j \in X$
(2) $j > \bigvee \{x \in L | x < j\}$
(3) j covers exactly one element, j_* and when $x < j$, we have $x \leq j_*$
(4) (If L is finite) $j \neq \widehat{0}$ and $j = x \vee y \implies j = x$ or $j = y$
(5) (If L is finite) j covers exactly one element

Definition. An element $j \in L$, where L is a complete lattice, is called ***complete join irreducible*** if it satisfies any and all of the first three conditions above. An element is called ***join irreducible*** if it satisfies (4), but this notion is not very useful in general. We sometime abbreviate complete join irreducible by ***cji***.

There is a dual notion of complete meet irreducible elements, defined in the analogous way, that we will not be needing at the moment.

Example. For S_3, using the Hasse Diagram, it is easy to see which elements are complete join irreducible, using condition (5) from the lemma. We just need to see which elements cover exactly one element:



Looking back the forcing diagram for S_3, note that each forcing equivalence class has exactly one cji pair (j_*, j) in it (but this uniqueness is not true in general!):

This suggests that maybe cji pairs could be in every forcing equivalence class, but before we get to that, we should prove this lemma.

Proof. (1) $\implies$ (2): If (2) is false, then $j = \bigvee\{x \in L | x < j\}$, and j is not in this set, which implies that (1) is false.

(2) $\implies$ (3): Let $j_* = \bigvee\{x \in L | x < j\} < j$. Note that j covers j_*, since $j_* < x < j$ implies $x \in \{x \in L | x < j\}$, so $x \leq j_*$. If j covers y, we have $y < j$, so $y \in \{x \in L | x < j\}$, so $y \leq j$, so $y = j_*$. And if $x < j$, then $x \in \{x \in L | x < j\}$, so $x \leq j_*$.

(3) $\implies$ (1): Suppose (1) is not true. Then we have some X with $\bigvee X = j$, and j is not in X. Note that $x < j$ for all $x \in X$, so $x \leq j_*$ for all $x \in X$, so $\bigvee X \leq j_* < j$, contradicting (3).

(1) $\implies$ (4) Note that $x \vee y = \bigvee\{x, y\}$, so if $x \vee y = j$, then $\bigvee x, y = j$, so $j \in \{x, y\}$.

(4) $\implies$ (5) (assuming L is finite): Let $j_* = \bigvee\{x \in L | x < j\}$. By repeated use of (4), we see that $j_* < j$. By the same reasoning as in (2) $\implies$ (3), we have that j covers only j_*.

(5) $\implies$ (3)(assuming L is finite): We already have most of (3), we just need to show that if $x < j$, then $x \leq j_*$. Take a chain $x = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_{m-1} \lessdot x_m = j$. Such a chain exists because L is finite. Now, j only covers one element, j_*, so we must have $x_{m-1} = j_*$. and then $x = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_{m-1} = j_*$ implies that $x \leq j_*$. □

Now, there is one standard result concerning cji elements in a lattice:

Lemma. Let L be a finite lattice. Then for $x \in L$, $x = \bigvee\{j \in L | j \text{ is cji and } j \leq x\}$

Proof. The proof is by induction. If $x = \widehat{0}$, then there are no *cji* elements less than it, so $\bigvee\{j \in L | j \text{ is cji and } j \leq x\} = \bigvee \emptyset = \widehat{0}$. If x is cji, then of course we have $x = x = \bigvee\{j \in L | j \text{ is cji and } j \leq x\}$, since $x \in x = \{j \in L | j \text{ is cji and } j \leq x\}$.

Now, if x is not cji, it covers at least two things, say $y_i \lessdot x$ for $i \in \{1, 2, \ldots, k\}$, $k \geq 2$. Now, $y_i = x = \bigvee\{j \in L | j \text{ is cji and } j \leq y_i\}$ by the inductive hypothesis, and we have $x = y_1 \vee \cdots \vee y_k = x = \bigvee\{j \in L | j \text{ is cji and } j \leq y_i, \text{for some } i\} = x = \bigvee\{j \in L | j \text{ is cji and } j \leq x\}$, since a cji element if less than x if and only if it is less than or equal to some element x covers. □

Now, we can prove what was hinted at before:

Lemma. Let L be a finite lattice. If $x \lessdot y$, then there exists a cji element $j \in L$ so that $(x, y) \iff (j_*, j)$.

Proof. Let $Z = \{z \in L | z \leq y, z \not\leq x\}$. Note that Z is nonempty, since $y \in Z$. Let j be the minimal element of Z.

I claim that j is cji. We know $j \neq \widehat{0}$, since $\widehat{0} \leq x$. With $j = u \vee v$ with $j \neq u, v$, then $u, v < j$, which means u, v are not in Z. Thus, we must have $u, v \leq x$ (as $u, v < j \leq y$), but then $j = u \vee y \leq x$, which is not true. So j must be cji.

Next, I claim that $j \vee x = y$ and $j \wedge x = j_*$. Note that $j \vee x \leq y$, and $j \vee x > x$ (since $j \vee x = x \implies j \leq x$). But $x \lessdot y$, so $j \vee x = y$. Similarly, $j \wedge x \geq j_*$, since j_* is not in Z, and $j_* \leq j \wedge x$ since $j_* \leq x, j$. Thus, $j_* = j \wedge x$.

Now, we have $x \lessdot y \implies j \wedge x = j_* \lessdot j = j \wedge y$. And also $j \lessdot j \implies x \vee j_* = x \lessdot y = x \vee j$. Thus $(x, y) \iff (j_*, j)$. ☐

In fact, we have proved something better than this. For two covers (x_1, y_1) and (x_2, y_2), we say that (x_1, y_1) ***slides to*** (x_2, y_2) if $y_1 = x_1 \vee y_2$ and $x_2 = x_1 \wedge y_2$. We write $(x_1, y_1) \rightsquigarrow (x_2, y_2)$. It is clear that $\rightsquigarrow$ is a partial order and that, if $(x_1, y_1) \rightsquigarrow (x_2, y_2)$ then $(x_1, y_1) \iff (x_2, y_2)$. So we have actually shown that, for all covers (x, y), there is a join irreducible element j with $(x, y) \rightsquigarrow (j_*, j)$ and, similarly, there is a meet irreducible element m with $(m, m^*) \rightsquigarrow (x, y)$.

We define two covers to be slide equivalent if we can get from one to the other by repeatedly sliding up and down.

October 30 and November 1 – Behavior of covers under quotients

On these days, we worked through a sequence of lemmas in IBL style concerning the relation between Covers(L) and Covers(L') for a lattice surjection $\pi : L \to L'$.

Defnitions: Let L be a finite lattice. We define Covers(L) to be $\{(x, y) \in L^2 : x \lessdot y\}$. We have natural inclusions JIrr(L) and MIrr$(L) \hookrightarrow$ Covers(L) by $j \mapsto (j_*, j)$ and $m \mapsto (m, m^*)$ respectively, and we will often abuse notation and think of JIrr(L) and MIrr(L) as subsets of Covers(L).

We define a binary relation $\implies$ called ***forcing*** on Covers(L) to be the transitive closure of the following conditions:

- If $x_1 \lessdot y_1$ and $x_1 \vee z \leq x_2 \lessdot y_2 \leq y_1 \vee z$ then $(x_1, y_1) \Longrightarrow (x_2, y_2)$.
- If $x_1 \lessdot y_1$ and $x_1 \wedge z \leq x_2 \lessdot y_2 \leq y_1 \wedge z$ then $(x_1, y_1) \Longrightarrow (x_2, y_2)$.

We define (x_1, y_1) and $(x_2, y_2) \in$ Covers(L) to be ***forcing equivalent*** if $(x_1, y_1) \iff (x_2, y_2)$.

For (x_1, y_1) and (x_2, y_2), we say that (x_1, y_1) ***slides to*** (x_2, y_2) if $x_1 \vee y_2 = y_1$ and $x_1 \wedge y_2 = x_2$. We write $(x_1, y_1) \rightsquigarrow (x_2, y_2)$. Define (x_1, y_1) and (x_2, y_2) to be ***slide equivalent*** if they are linked by a chain of slides and reverse slides. So slide equivalent elements are forcing equivalent, but perhaps not vice versa.

Let L and L' be finite lattices and $\pi : L \to L'$ a lattice **surjection**. Let E be the set of $(x, y) \in$ Covers(L) with $\pi(x) = \pi(y)$.

Problem 1: Suppose that x and $y \in L$ with $x \lessdot y$ and $\pi(x) \neq \pi(y)$. Show that $\pi(x) \lessdot \pi(y)$.
Thus, π induces a map $\pi_* :$ Covers$(L) \setminus E \to$ Covers(L') with $\pi_*(x, y) = (\pi(x), \pi(y))$.

Proof. Suppose to the contrary that $\pi(x) < z' < \pi(y)$. Lift z' to z in L. We know that $x \leq y \wedge (x \vee z) \leq y$, so either $y \wedge (x \vee z) = x$ or $y \wedge (x \vee z) = y$ since $x \lessdot y$. Next, notice that

$$\pi(y \wedge (x \vee z)) = \pi(y) \wedge (\pi(x) \vee \pi(z)) = \pi(y) \wedge (\pi(x) \vee z') = \pi(y) \wedge z' = z'.$$

Therefore, we know that $z' = \pi(y \wedge (x \vee z)) = \pi(x)$ or $\pi(y)$, which is a contradiction, as we initially claimed that $\pi(x) < z' < \pi(y)$. Thus, we can conclude that $\pi(x) \lessdot \pi(y)$. ☐

Problem 2: Let $x' \leq y'$ in L'. Show that we can find x and $y \in L$ with $x \leq y$, $\pi(x) = x'$ and $\pi(y) = y'$.

Proof. Let u and $v \in L$ be lifts of x' and $y' \in L'$ respectively. Let $x = u \wedge v$ and $y = u \vee v$. Then $x \leq y$, We have $\pi(x) = \pi(u \wedge v) = x' \wedge y' = x'$ and $\pi(y) = \pi(u \vee v) = x' \vee y' = y'$. □

Problem 3: Let $x' \lessdot y'$ in L'. Show that we can find x and $y \in L$ with $x \lessdot y$, $\pi(x) = x'$ and $\pi(y) = y'$.

In other words, Problem 3 shows that π_* is surjective.

Proof. Note that since $x' \lessdot y'$, we have $x' \leq y'$, so by problem 2, we can find $x, y \in L$ with $\pi(x) = x'$, $\pi(y) = y'$ and $x < y$ (since $\pi(x) = x' < y' = \pi(y)$, the inequality must be strict). Now fix a path of covers from x to y. Since $\pi(x) = x' \lessdot y' = \pi(y)$, everything on the path must map to one of x', y' under π, and there must be a cover $x_0 \lessdot y_0$ with $\pi(x_0) = x'$ and $\pi(y_0) = y'$. Hence, (x_0, y_0) is the desired cover in L. □

Problem 4: Suppose that (x_1, y_1) and $(x_2, y_2) \in \mathrm{Covers}(L) \setminus E$. Show that, if $(x_1, y_1) \rightsquigarrow (x_2, y_2)$ then $(\pi(x_1), \pi(y_1)) \rightsquigarrow (\pi(x_2), \pi(y_2))$. Conclude that π_* descends to a well defined map from $\mathrm{Covers}(L) \setminus E$ modulo slide equivalence to $\mathrm{Covers}(L')$ modulo slide equivalence.

Proof. Since $(x_1, y_1) \rightsquigarrow (x_2, y_2)$, $x_1 \vee y_2 = y_1$ and $x_1 \wedge y_2 = x_2$. Then since π is a lattice homomorphism, $\pi(x_1) \vee \pi(y_2) = \pi(y_1)$ and $\pi(x_1) \wedge \pi(y_2) = \pi(x_2)$. We also know from Problem 1 that $\pi(x_1) \lessdot \pi(y_1)$ and $\pi(x_2) \lessdot \pi(y_2)$, so this does give a slide $(\pi(x_1), \pi(y_1)) \rightsquigarrow (\pi(x_2), \pi(y_2))$.

To infer that π_* is thus a well-defined map as required, it remains only to show that a chain of slides and reverse slides defining a slide equivalence between two covers in $\mathrm{Covers}(L) \setminus E$ cannot pass through a cover in E (and thus descends to a chain of slides in $\mathrm{Covers}(L')$.) If we had a slide $(x_1, y_1) \rightsquigarrow (x_2, y_2)$ with $(x_2, y_2) \in E$, then (x_2, y_2) forces (x_1, y_1) since $x_1 = x_2 \vee x_1$ and $y_1 = y_2 \vee x_1$. This implies $(x_1, y_1) \in E$, a contradiction. The same issue occurs with meet instead of join if $(x_1, y_1) \in E$. □

Problem 5: Suppose that (x_1, y_1) and $(x_2, y_2) \in \mathrm{Covers}(L) \setminus E$ with $\pi(x_1) = \pi(x_2)$ and $\pi(y_1) = \pi(y_2)$. In this problem, you will show that (x_1, y_1) and (x_2, y_2) are slide equivalent.

(a) Suppose that $y_1 \geq y_2$. Show that $x_1 \geq x_2$. Show furthermore that, in this case, $(x_1, y_1) \rightsquigarrow (x_2, y_2)$.

(b) Now assume only that (x_1, y_1) and $(x_2, y_2) \in \mathrm{Covers}(L) \setminus E$ with $\pi(x_1) = \pi(x_2)$ and $\pi(y_1) = \pi(y_2)$. Show that (x_1, y_1) and (x_2, y_2) are slide equivalent.

Proof. Suppose that (x_1, y_1) and $(x_2, y_2) \in \mathrm{Covers}(L) \setminus E$ with $x' := \pi(x_1) = \pi(x_2)$ and $y' := \pi(y_1) = \pi(y_2)$. We wish to show that (x_1, y_1) and (x_2, y_2) are slide equivalent.

(a) Suppose that $y_1 \geq y_2$. We claim that $x_1 \geq x_2$. To see this, note that $x_1 \leq x_1 \vee x_2 \leq y_1 \vee y_2 = y_1$, so since (x_1, y_1) is a cover, either $x_1 \vee x_2 = x_1$ or $x_1 \vee x_2 = y_1$. But since $(x_1, y_1) \notin E$, we have $\pi(x_1 \vee x_2) = \pi(x_1) \vee \pi(x_2) = x' < y' = \pi(y_1)$, so that $x_1 \vee x_2 = x_1$. Thus $x_1 \geq x_2$.

Moreover, in this case, we have that $x_1 \leq x_1 \vee y_2 \leq y_1$, so since (x_1, y_1) is a cover, $x_1 \vee y_2 \{x_1, y_1\}$. Since $\pi(x_1 \vee y_2) = \pi(x_1) \vee \pi(y_2) = x' \vee y' = y' \neq \pi(x_1)$, we must have $x_1 \vee y_2 = y_1$.

Similarly, note that $x_2 \leq x_1 \wedge y_2 \leq y_2$, so since (x_2, y_2) is a cover, $x_1 \wedge y_2 \in \{x_2, y_2\}$. Since $\pi(x_1 \wedge y_2) = \pi(x_1) \wedge \pi(y_2) = x' \wedge y' = x' \neq \pi(y_2)$, we must have $x_1 \wedge y_2 = x_1$. Thus if $y_1 \geq y_2$, then (x_1, y_1) slides to (x_2, y_2).

(b) We claim that without the hypothesis that $y_1 \leq y_2$, (x_1, y_1) and (x_2, y_2) are still slide equivalent. To see this, let

$$x_1 \wedge x_2 = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_m = y_1 \wedge y_2$$

be a chain of covers connecting $x_1 \wedge x_2$ to $y_1 \wedge y_2$. Note that since

$$x' = \pi(z_0) \leq \pi(z_1) \leq \cdots \leq \pi(z_m) = y',$$

there exists some $k \in \{1, \cdots, m\}$ such that $\pi(z_i) = x'$ for all $i < k$ and $\pi(z_i) = y'$ for all $i \geq k$. Moreover, note that since $z_k \leq y_1 \wedge y_2 \leq y_1$, we have that $y_1 \geq z_k$ and $y_2 \geq z_k$. Thus the cover (z_{k-1}, z_k) satisfies the hypotheses of (a), so it is slide equivalent to both (x_1, y_1) and (x_2, y_2). It follows that (x_1, y_1) and (x_2, y_2) are slide equivalent. □

Problem 6: Suppose that (x_1', y_1') and $(x_2', y_2') \in \mathrm{Covers}(L)$ and $(x_1', y_1') \rightsquigarrow (x_2', y_2')$. In this problem, we will show that we can lift x_1', y_1', x_2', y_2' to x_1, y_1, x_2, y_2 with $(x_1, y_1) \rightsquigarrow (x_2, y_2)$.

(a) Show that we can lift x_1', y_1', x_2', y_2' to x_1, y_1, x_2, y_2 with $x_1 \lessdot y_1$, $x_2 \lessdot y_2$ and $y_1 \geq y_2$.
(b) Show that, if we choose lifts as in part (a), we will also have $x_1 \geq x_2$ and $(x_1, y_1) \rightsquigarrow (x_2, y_2)$.

Proof. (a) By problem 3, we can lift the covers (x_1', y_1') and (x_2', y_2') of L' to covers (x_1, y_1) and (x_2, y_2) of L. Define $y = y_1 \vee y_2$, and $x = x_1 \vee x_2$. Then note that $\pi(y) = \pi(y_1 \vee y_2) = y_1' \vee y_2' = y_1'$, where the last equality comes from the fact that $y_1' = x_1' \vee y_2'$ because $(x_1', y_1,)$ slides to (x_2', y_2'). Similarly, $\pi(x) = \pi(x_1 \vee x_2) = x_1' \vee x_2' = x_1'$, where the last equality comes from the fact that $x_1' \vee y_2' = x_2'$.

We do not necessarily have that (x, y) is a cover, but we can find a chain $x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_k = y$. Since $\pi(x) = x_1'$, and $\pi(y) = y_1'$, and $x_1' \lessdot y_1'$, we must have some index where $\pi(z_i) = x_1'$ and $\pi(z_{i+1}) = y_1'$.

We claim that $y_2 \leq z_{i+1}$. This is true because $z_{i+1} \geq x_1 \vee x_2 \geq x_2$, and so $x_2 \leq z_{i+1} \wedge y_2 \leq y_2$. Since (x_2, y_2) is a cover, $z_{i+1} \wedge y_2 = y_2$ or $z_{i+1} \wedge y_2 = x_2$. But $\pi(z_{i+1} \wedge y_2) = y_1' \wedge y_2' = y_2'$, so $z_{i+1} \wedge y_2 = y_2$, so $y_2 \leq z_{i+1}$.

We can replace (x_1, y_1) with (z_i, z_{i+1}) to get the desired properties: (z_i, z_{i+1}) is a cover, $\pi(z_i) = x_1'$, $\pi(z_{i+1}) = y_1'$, and $z_{i+1} \geq y_2$ as described above.

(b) From part (a) we can assume we have lifted (x_1', y_1') and (x_2', y_2') to covers (x_1, y_1) and (x_2, y_2) so that $y_1 \geq y_2$. Now we want to tweak things to have $x_1 \geq x_2$. Consider $y_2 \wedge x_1$. Note that $\pi(y_2 \wedge x_1) = y_2' \wedge x_1' = x_2'$. Now, consider a chain $y_2 \wedge x_1 = z_0 \lessdot \cdots \lessdot z_k = y_2$. Since $\pi(y_2) = y_2'$ and $\pi(y_2 \wedge x_1) = x_2'$, we must have some i where $\pi(z_i) = x_2'$ and $\pi(z_{i+1}) = y_2'$.

I claim that $z_i \leq x_1$. Note $z_i \leq z_k = y_2 \leq y_1$, and so $x_1 \leq z_i \vee x_1 \leq y_1$. Since (x_1, y_1) is a cover, we must have $z_i \vee x_1 = x_1$ or $z_i \vee x_1 = y_1$. But note $\pi(z_i \vee x_1) = x_2' \vee x_1' = x_1$ which means that we need $z_i \vee x_1 = x_1$, or $z_i \leq x_1$.

Now we can replace (x_2, y_2) with (z_i, z_{i+1}) to get the desired properties: $(z_i, z_{i+1}$ is a cover, $\pi(z_i) = x_2'$, $\pi(z_{i+1}) = y_2'$, $z_{i+1} \leq y_2 \leq y_1$, and $z_i \leq x_1$. □

Combining Problems 5 and 6, π_* is a bijection from $\mathrm{Covers}(L) \setminus E$ modulo slide equivalence to $\mathrm{Covers}(L')$ modulo slide equivalence.

Problem 7: Suppose that (x_1, y_1) and $(x_2, y_2) \in \mathrm{Covers}(L) \setminus E$. Show that, if $(x_1, y_1) \Longrightarrow (x_2, y_2)$ then $(\pi(x_1), \pi(y_1)) \Longrightarrow (\pi(x_2), \pi(y_2))$.

Proof. We know there exists some $z \in L$ such that (without loss of generality, up to turning over the $\vee$'s) $x_1 \vee z \leq x_2 \lessdot y_2 \leq y_1 \vee z$. Then since π is a lattice homomorphism, we have

$$\pi(x_1) \vee \pi(z) \leq \pi(x_2) \lessdot \pi(y_2) \leq \pi(y_1) \vee \pi(z)$$

which shows that $(\pi(x_1), \pi(y_1)) \Rightarrow (\pi(x_2), \pi(y_2))$. □

Problem 8: Suppose that (x_1, y_1) and $(x_2, y_2) \in \mathrm{Covers}(L) \backslash E$. Show that, if $(\pi(x_1), \pi(y_1)) \Longrightarrow (\pi(x_2), \pi(y_2))$ then $(x_1, y_1) \Longrightarrow (x_2, y_2)$.

In other words, the forcing preorder on $\mathrm{Covers}(L')/(\rightsquigarrow$ equivalence) is precisely the preorder induced on $\mathrm{Covers}(L')/(\rightsquigarrow$ equivalence). considered as a subset of $(\mathrm{Covers}(L) \backslash E)/(\rightsquigarrow$ equivalence).

Proof. We prove the contrapositive. Suppose that (x_1, y_1) does not force (x_2, y_2). Let E' be the set of covers forced by (x_1, y_1). Then $E \cup E'$ is closed under forcing. Let L'' be the quotient of L by $E \cup E'$. Since $L \to L''$ collapses E, the map $p : L \to L''$ factors through $\pi : L \to L'$ as a map of sets. Let $\pi' : L' \to L''$ be the induced map. We claim that π' is a lattice map. Indeed, let x' and $y' \in L'$ and lift them to x and $y \in L$. Then $\pi'(x' \wedge y') = \pi'(\pi(x) \wedge \pi(y)) = \pi'(\pi(x \wedge y)) = p(x \wedge y) = p(x) \wedge p(y) = \pi'(\pi(x)) \wedge \pi'(\pi(y)) = \pi'(x') \wedge \pi'(y')$ using that p and π are maps of lattices and $p = \pi' \circ \pi$. The argument for $\vee$ is similar.

So π' is a lattice homorphism which contracts $(\pi(x_1), \pi(y_1))$ and not $(\pi(x_2), \pi(y_2))$, and thus $(\pi(x_1), \pi(y_1))$ does not force $(\pi(x_2), \pi(y_2))$. □

November 4 – Congruence uniform lattices and doubling, part 1

Our big goal is to understand quotients of lattices. Our starting point was a result of Day

Theorem. (Day) A lattice quotient is determined by which covers it contracts. The contractible sets E are sets closed under the forcing relation, $\Longrightarrow$.

The forcing relation is hard to deal with. We introduced a more convenient relation, sliding, which puts a partial order on $\mathrm{Covers}(L)$. We showed that, for every cover (x, y), there is at least one join irreducible j and at least one meet irreducible m with $(m, m^*) \rightsquigarrow (x, y) \rightsquigarrow (j_*, j)$. We now want to study the cases where this j and m are unique. We could ask for two things:

Definition. (Stronger) A lattice L is ***congruence uniform*** or CU if every $\Leftrightarrow$ equivalence class has exactly one (j_*, j) pair and exactly one pair $(m, m*)$ pair.

Definition. (Weaker) A lattice ***has good*** $\rightsquigarrow$ ***equivalence representatives*** if each slide equivalence class has a unique (j_*, j) and a unique $(m, m*)$ cover.

We will show having good $\rightsquigarrow$ equivalence representatives is equivalent to a very different sounding condition, semi-distributivity.

Today, we focus on congruence uniformity. Here is the main result we are heading for:

Theorem. (Day[14]) A finite lattice L is CU iff it can be produced from the trivial lattice by a sequence of interval doublings, i.e there exists $L \cong L_N \twoheadrightarrow L_{N-1} \twoheadrightarrow \cdots \twoheadrightarrow L_1 \twoheadrightarrow L_0 = \{pt\}$, each $L_k \twoheadrightarrow L_{k-1}$ is a doubling.

Of course, this means that we have to define doubling.

Let L' be a lattice and let $I' = [a', b']$ be an interval of L'.

As a set, we define the ***doubling*** $L'[I']$ to be $(L' - I') \sqcup (I' \times \{0, 1\})$. Let $\pi : L'[I'] :\to L'$ be the obvious projection. For x and $y \in L'[I']$, let $x \leq y$ if:

(1) $\pi(x) \leq \pi(y)$.
(2) If $x = (\pi(x), i)$ and $y = (\pi(y), j)$ are in $I' \times \{0, 1\}$, we also require that $i \leq j$.

[14]Alan Day, "Characterizations of finite lattices that are bounded-homomorphic images of sub- lattices of free lattices", *Canad. J. Math.* **31** (1979), no. 1, 69–78, Corollary 5.4.

The following claims will appear on homework:

(1) $L'[I']$ is a lattice.
(2) $\pi : L'[I'] \to L'$ is a lattice quotient.
(3) The collapsed edges of π are precisely $(x, 0) \lessdot (x, 1)$. Exactly one of these is join irreducible, namely, $(a', 0) \lessdot (a', 1)$. Exactly one of these is meet irreducible, $(b', 0) \lessdot (b', 1)$, where $I' = [a', b']$.

Let's verify that the doubling of a CU lattice is CU, thus checking the easy part of Day's result. From our worksheets on the behavior of forcing under lattice quotients, The forcing equivalence classes of $L'[I']$ are $\{(x, 0) \lessdot (x, 1)\}$ and π_*^{-1} of forcing classes of L'. Clearly, the former class has unique (j_*, j) and unique (m, m^*) covers. We need to check that the latter does as well.

In other words, let C' be a forcing equivalence class of L' containing a unique (j'_*, j') and a unique $(m', (m')^*)$. Let $C = \pi^{-1}(C')$. We need to check that there is a unique join irreducible cover in C.

Let j_0 be the bottom element of $\pi^{-1}(j')$. We claim that

(1) (j'_*, j') lifts to a (k, j_0) cover.
(2) j_0 is join irreducible.
(3) No other (j_*, j) lift exists.

Proof. Proof of (2): Suppose for the sake of contradiction, $j_0 = x \vee y, x, y \leq j_0$. Note that $\pi(x) \neq \pi(y)$, as otherwise $\pi(j_0) = \pi(x) \vee \pi(y)$ would be $\pi(x) < \pi(j_0)$. So $\pi(j_0) = \pi(x) \vee \pi(y)$ and $\pi(x), \pi(y) < \pi(j_0)$, which is a contradiction.

Proof of (1): Lift (j'_*, j') to $j_2 < j_1$, replace by $j_0 \wedge j_2 < j_0 \wedge j_1 = j_0$. Take chain of covers

$$j_0 \wedge j_2 \lessdot \cdots \lessdot j_0$$

$j_0 \wedge j_2$ maps to j'_*, and j_0 maps to j'. By minimality of j_0, the last cover in the chain is the desired one.

Proof of (3): If (x, y) is another lift, then $y \geq x, j_0$. So $j_0 \vee x = x$ or y. Since $\pi(x) = j'_* < j'$, so $j_0 \vee x \neq x$, so $j_0 \vee x = y$. So y is not join irreducible. ∎

So doubling preserves the property of being CU and we have the easy direction of Day's result. Next time, we prove the hard part.

November 6 – Congruence uniform lattices and doubling, part 2

Let L be a finite lattice. We now want to show that, if L is congruence uniform, then L is obtained from the trivial lattice by a sequence of doublings. We repeat that L is defined to be congruence uniform (CU) if every forcing equivalence has a unique join irreducible cover and a unique meet irreducible cover. Since forcing equivalence is defined as the equivalence classes of the forcing preorder, we can totally order the forcing equivalence classes as C_1, C_2, ..., C_N such that, if $C_i \Longrightarrow C_j$ then $i \leq j$. Then we have a sequence of quotients

$$L \twoheadrightarrow L/C_N \twoheadrightarrow L/(C_N \cup C_{N-1}) \twoheadrightarrow \cdots \twoheadrightarrow L/(C_1 \cup C_2 \cup \cdots \cup C_N) \cong \{0\}.$$

By our results on forcing and slide equivalence under quotients, each step $L_{k+1} \twoheadrightarrow L_k$ in this chain collapses a single forcing equivalence class of L_{k+1} and that class has a unique join irreducible cover and a unique meet irreducible cover. Thus we are reduced to studying the situation that $\pi : L \to L'$ is a lattice surjection collapsing a single join irreducible cover

(j_*, j) and a single meet irreducible cover (m, m^*). We want to show that $L \cong L'[I']$ for some interval I' of L'; then Day's result will follow.

Let E be the set of covers collapsed by π. We first want:

Lemma. Each $x \in L$ is adjacent to at most one edge e in E.

Proof. We must rule out the following three cases:

Case 1 There is an element x with two covers $x \lessdot y_1$ and $x \lessdot y_2$ in E. Then $y_1 = x \vee j = y_2$.

Case 2 There is an element y with two covers $x_1 \lessdot y$ and $x_1 \lessdot y$ in E. Then $x_1 = y \wedge m = x_2$.

Case 3 There are covers $x \lessdot y \lessdot z$ in E. Then the relations $(m, m^*) \rightsquigarrow (y, z)$ and $(x, y) \rightsquigarrow (j_*, j)$ imply that $m \geq y \geq j$, but this contradicts that $(m, m^*) \rightsquigarrow (j_*, j)$. □

So each fiber of π has either 1 or 2 elements.

Let $I' = [\pi(j), \pi(m)]$. We will be showing that $L \cong L'[I']$ (and the map $L \to L'$ is the standard map $L'[I'] \to L'$.

We start by showing that π is 2 to 1 above $[\pi(j), \pi(m)]$ and 1 to 1 everywhere else. For the first claim, suppose that $\pi(j) \leq x' \leq \pi(m)$ and $\pi^{-1}(x')$ is the singleton $\{x\}$. Then $m \geq x \geq j$, contradicting that $(m, m^*) \rightsquigarrow (j, j_*)$. For the second claim,If $\pi(x) = \pi(y)$ with $x \lessdot y$, then $(m, m^*) \rightsquigarrow (x, y) \rightsquigarrow (j_*, j)$ so $m \geq x$ and $y \geq j$, and we get $\pi(m) \geq \pi(y) = \pi(x) \geq \pi(j)$.

We next claim that the order of L on $\pi^{-1}(I')$ is $I' \times \{0, 1\}$. Let $z' \in I'$ and let z be an arbitrary preimage of z. Put $x = z \wedge m$ and $y = z \vee j$. Then $\pi(x) = \pi(z) \wedge \pi(m) = z' \wedge \pi(m) = z'$ and similarly $\pi(y) = z'$. Also, we have $x \neq y$ since, if $x = y$, then $m \geq x = y \geq j$, a contradiction. So we have constructed two distinct elements of $\pi^{-1}(z')$ and we must have $\pi^{-1}(z') = \{x, y\}$. Thus, one of the preimages of z is in $[j_*, m]$ (namely, x) and the other is in $[j, m^*]$. Call these preimages $(z, 0)$ and $(z, 1)$. We have thus made a bijection between $\pi^{-1}(I')$ and $I' \times \{0, 1\}$; we claim this bijection is order preserving. Clearly, if $(x, r) \leq (y, s)$, we must have $\pi(x) \leq \pi(y)$. I also claim we must have $r \leq s$: Indeed, we cannot have $(x, 1) \leq (y, 0)$, as $m \geq (y, 0)$ and $(x, 1) \geq j$. Conversely, I claim that, if $x' \leq y'$ in I' and $r \leq s$ in $\{0, 1\}$, then $(x, r) \leq (y, s)$. If $r = s = 0$, take lifts $\tilde{x} \leq \tilde{y}$. Then $(x, 0) = m \wedge \tilde{x} \leq m \wedge \tilde{y} = (y, 0)$. The analogous argument works for $r = s = 1$. Finally, if $r = 0$ and $s = 1$, just notice that $(x, 0) \leq (y, 0) < (y, 1)$.

It now remains to check that the induced order on the rest of L matches that on $L'[I']$. This is tedious but straightforward. Let x and $y \in L$ and not both in $\pi^{-1}(I')$. If $x \leq y$ in L then $\pi(x) \leq \pi(y)$ in L' so $x \leq y$ in $L'[I']$. We now must assume that $x \leq y$ in $L'[I']$ and prove this inequality holds in L.

So, let x and $y \in L$ and not both in $\pi^{-1}(I')$, with $x \leq y$. Then $\pi(x) \leq \pi(y)$. We can lift $\pi(x)$ and $\pi(y)$ to some $\tilde{x}$ and $\tilde{y}$ in L with $\tilde{x} \leq \tilde{y}$. For the elements not in I', the lift is unique. Thus, if $x \notin I'$ and $y \notin I'$, we must have $x \leq y$.

Suppose now that $x \notin I'$ but $y = (y', r) \in I'$. Then we have shown that $x \leq (y', s)$ for some s, but not necessarily the same one. Since $(y', 0) < (y', 1)$, the only concern is that we might have $x \leq (y', 1)$ but $x \not\leq (y', 0)$. But then consider $x_2 := x \wedge (y, 0)$. We have $\pi(x_2) = \pi(x) \wedge \pi(y, 0) = x' \wedge y' = x'$, so x_2 is a preimage of x'. Since $x' \notin I'$, we must have $x_2 = x$. But then $x \leq (y', 0)$ as required. The case where $x \in I'$ and $y \notin I'$ is similar.

November 8 – Semidistributivity

Let L be a finite lattice. We said that L has ***good slide representatives*** if, for every cover (x, y) there are unique $j \in \mathrm{JIrr}$ and $m \in \mathrm{MIrr}$ with $(m, m^*) \rightsquigarrow (x, y) \rightsquigarrow (j_*, j)$. Today, we are going to show that this is equivalent to a seemingly very different condition.

We say that a lattice L is ***semidistributive*** (SD) if

(1) If $x \vee z_1 = x \vee z_2 = y$ then $x \vee (z_1 \wedge z_2) = y$ and
(2) if $y \wedge z_1 = y \wedge z_2 = x$ then $y \wedge (z_1 \vee z_2) = x$.

Semidistributivity is a weakening of a more natural condition called distributivity: A lattice is ***distributive*** if $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. In fact, these two conditions are equivalent: If we assume the first, then we have

$$(p \wedge q) \vee (p \wedge r) = (p \wedge p) \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r) = p \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r) = p \vee (q \wedge r)$$

since $p \geq p \wedge q$, $p \wedge r$.

To see why distributivity implies semidistributivity note that, if L is distributive and $x \vee z_1 = x \vee z_2 = y$ then $x \vee (z_1 \wedge z_2) = (x \vee z_1) \wedge (x \vee z_2) = y \wedge y = y$, and the other condition is similar.

Today's main result is

Theorem. Let L be a finite lattice. Then L is SD iff it has good slide equivalent representatives.

In class I attributed this to Day, but that appears to be wrong. Rather, this seems to be one of the results that experts knew for a long time, but where it isn't clear who did it first. The earliest reference I found is *Free Lattices*, by Freese, Jezek and Nation, Mathematical Surveys and Monographs 1995 Volume 42, Chapter 2.5.

Let's first show that, if L is semidistributive, then L has good slide representatives. Let (x, y) be a cover of L. Recall that, when we proved that (x, y) slides to a join irreducible cover, we introduced the set $Z := \{z \in L : x \wedge z = y\}$. We showed then that, if j is minimal in Z, then j is join irreducible and $(x, y) \rightsquigarrow (j_*, j)$.

It turns out that the converse also holds, and we will need this today. Indeed, suppose that j is join irreducible and $(x, y) \rightsquigarrow (j_*, j)$. By the definition of sliding, $j \in Z$. Moreover, if $w < j$ then $w \leq j_*$ so $x \wedge w \leq x \wedge j_* = x < y$, so we see that j is minimal in Z.

So our goal, is to show that Z has a unique minimal element, using the SD hypothesis. Indeed, (half of) the definition of SD is precisely that Z is closed under meet, so $\bigwedge Z \in Z$ and is the unique minimal element. This shows that (x, y) slides to a unique join irreducible cover, and we similarly show that (x, y) is slid to by a unique meet irreducible cover.

Now let's show the converse: **If L has good slide representatives, then L is SD.** Suppose that $y \wedge z_1 = y \wedge z_2 = x$; we want to show $y \wedge (z_1 \vee z_2) = x$. We clearly have $y \wedge (z_1 \vee z_2) \geq x$. If they are not equal, choose chain of covers: $x \lessdot y' \lessdot \cdots \lessdot y \wedge (z_1 \vee z_2)$, as depicted in figure 20.

We first claim that we have $y' \not\leq z_1$ and $y' \not\leq z_2$. Indeed, if $y' \leq z_j$, then y' is a lower bound of y and z_j, but $y' > x = y \wedge z_j$, a contradiction.

So $x = y' \wedge z_1 = y' \wedge z_2$. Our assumption of good slide representatives shows that $\{z : y' \wedge z = x\}$ has a maximal element, and thus we must have $y' \wedge (z_1 \vee z_2) = x$. But $y' \wedge (z_1 \vee z_2) = y'$, a contradiction. Λ

We close with a lemma we'll want on Monday. This is taken from Nathan Reading, "Lattice Theory of the Poset of Regions", Chapter 9 of *Lattice Theory: Special Topics and Applications, Volume 2*.

Lemma. (Monday's lemma) Let L be finite lattice. Suppose that

(1) whenever $w_1, w_2 \gtrdot w_1 \wedge w_2$ and $x = y \wedge w_1 = y \wedge w_2$, then $x = y \wedge (w_1 \vee w_2)$, and
(2) whenever $w_1, w_2 \lessdot w_1 \vee w_2$ and $x = y \vee w_1 = y \vee w_2$, then $x = y \vee (w_1 \wedge w_2)$

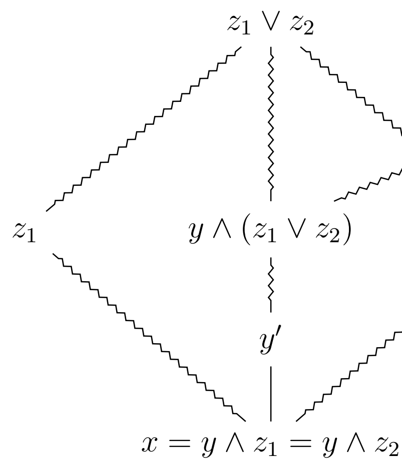


FIGURE 20. Deducing semidistributivity from good slide representatives

Then l is SD.

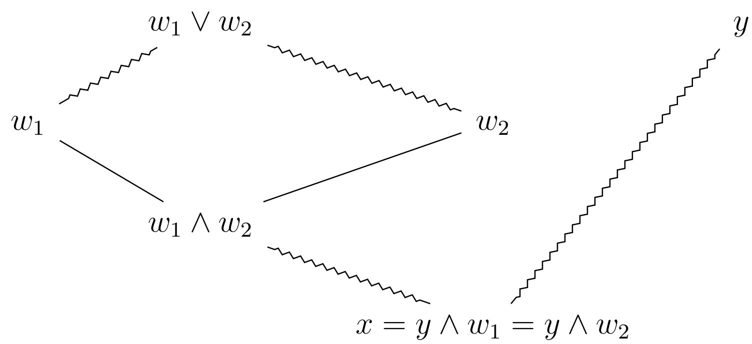


FIGURE 21. Condition (1) of the lemma

Proof. We will verify that, if $y \wedge z_1 = y \wedge z_2 = x$ then $y \wedge (z_1 \vee z_2) = x$. Our proof is by induction on the distance from $z_1 \wedge z_2$ to $\hat{1}$. For the base case, if $z_1 \wedge z_2 = \hat{1}$ then $z_1 = z_2 = \hat{1}$ as well, and the statement is clear.

Let $x = y \wedge z_1 = y \wedge z_2$ and note that we also have $x = y \wedge z_1 \wedge z_2$. Choose chains from z_1 down to $z_1 \wedge z_2$ and z_2 down to $z_1 \wedge z_2$. Let $w_1 \gtrdot z_1 \wedge z_2$ and $w_2 \gtrdot z_1 \wedge z_2$ be the last step of each chain. The next steps are depicted in Figure 22.

Our hypothesis is that $y \wedge (w_1 \vee w_2) = x$, since w_1 and w_2 cover $w_1 \wedge w_2$. Also recall that $y \wedge z_1 = x$ and $y \wedge z_2 = x$.

We have $z_1 \wedge (w_1 \vee w_2) \geq w_1 > z_1 \wedge z_2$ so, inductively, we have $y \wedge (z_1 \vee w_1 \vee w_2) = x$. Similarly, $y \wedge (z_2 \vee w_1 \vee w_2) = x$. Then $(z_1 \vee w_1 \vee w_2) \wedge (z_2 \vee w_1 \vee w_2) \geq w_1 \vee w_2 > z_1 \wedge z_2$ so induction also gives $y \wedge ((z_1 \vee w_1 \vee w_2) \vee (z_2 \vee w_1 \vee w_2))$. But $((z_1 \vee w_1 \vee w_2) \vee (z_2 \vee w_1 \vee w_2)) = z_1 \vee z_2 \vee w_1 \vee w_2 = z_1 \vee z_2$, so we have shown $y \wedge (z_1 \vee z_2) = x$ as desired. ☐

NOVEMBER 11 – WEAK ORDER IS SEMIDISTRIBUTIVE

Today, we finally return to Coxeter groups! Our goal for today is to prove:

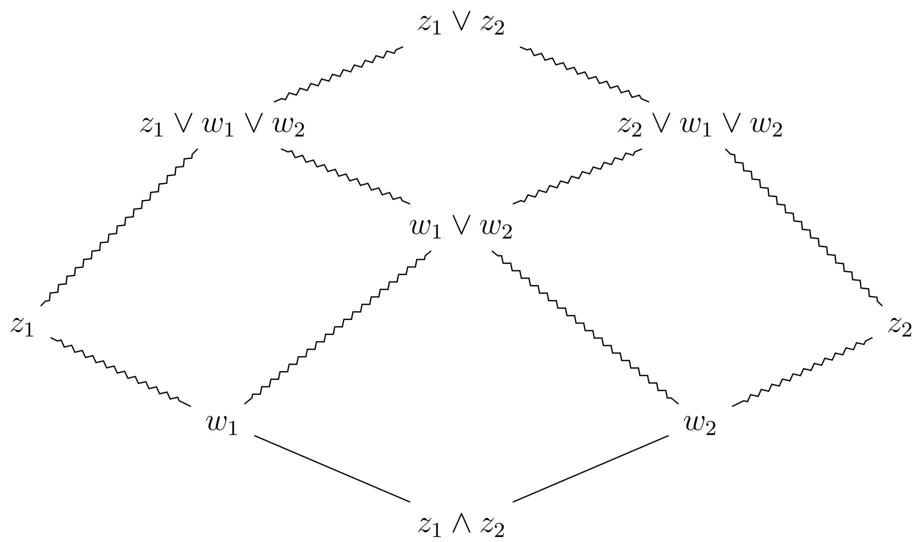


FIGURE 22. Proof of lemma

Theorem. For W any Coxeter group and $a \in W$, the interval $[e, a]$ is a semidistributive lattice.

Remark. Once we have this, we'll also know that this is true of any interval in the Coxeter group, since sublattices of semidistributive lattices are semidistributive.

Remark. In fact, these intervals are also congruence uniform lattices, but this will take longer to prove, and we'll do it later.

Proof. It should come as no surprise that we'll use a Björner-Edelman-Ziegler-style lemma to prove this, as that's how we showed these intervals are lattices in the first place, and we also proved such a lemma at the end of the previous class. Recall:

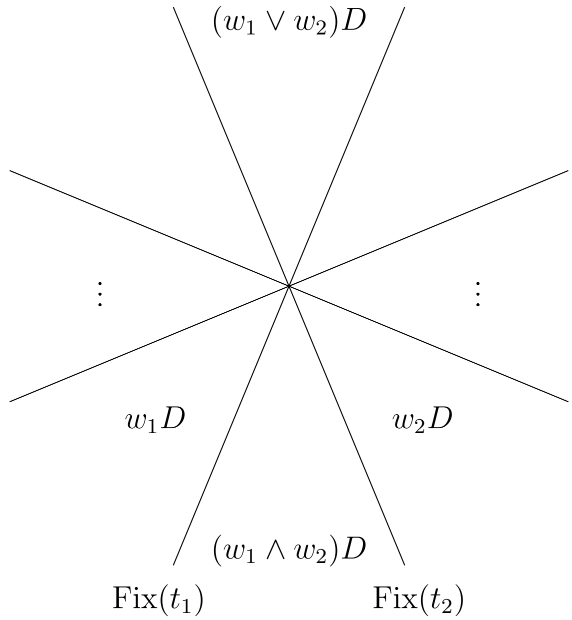
Lemma. Suppose L is a lattice. Suppose that:

- whenever $w_1, w_2 \gtrdot w_1 \wedge w_2$ and $y \wedge w_1 = y \wedge w_2 = x$, we have $y \wedge (w_1 \vee w_2) = x$
- dually, whenever $w_1, w_2 \lessdot w_1 \vee w_2$ and $x \vee w_1 = x \vee w_2 = y$, we have $x \vee (w_1 \wedge w_2) = y$

Then L is semidistributive (i.e., the first hypothesis in each bullet point is unnecessary)

Let's apply this to right order on a Coxeter group. We'll just check the first bullet point in the lemma, because the second is similar.

So suppose that $w_1, w_2 \gtrdot w_1 \wedge w_2$. Then, letting D be the usual fundamental domain in $V^\vee$, the regions $w_1 D$ and $w_2 D$ are both separated from $(w_1 \wedge w_2)D$ by single hyperplanes. Looking at the hyperplane arrangement, we're in a situation that looks roughly like this (though perhaps with more than 4 hyperplanes):



Let $\mathrm{Fix}(t_1)$ and $\mathrm{Fix}(t_2)$ be the hyperplanes separating w_1D and w_2D (respectively) from the region $(w_1 \wedge w_2)D$.

We note first that $w_1 \vee w_2$ actually exists in the interval $[e, a]$, and is the same as in the full lattice of L, because a is an upper bound for w_1 and w_2. This is good, because the hypotheses of the lemma wouldn't make sense otherwise.

Now consider y and x such that $y \wedge w_1 = y \wedge w_2 = x$. This additionally implies that $y \wedge (w_1 \wedge w_2) = x$. We then have $y \wedge (w_1 \vee w_2) \geq x$, and we want to show that this is an equality.

Suppose for a contradiction that it is not. Then choose some x' with $y \wedge (w_1 \vee w_2) \geq x' \gtrdot x$, and let $\mathrm{Fix}(t)$ be the hyperplane separating xD and $x'D$. We'll now narrow down where this hyperplane must be.

We know that $x' \not\leq w_1 \wedge w_2$, because if this inequality did hold, x' would be a lower bound of both y and $w_1 \wedge w_2$ greater than x, contradicting that $y \wedge (w_1 \wedge w_2) = x$. Thus there is some reflection in W which is an inversion of x', but not of $w_1 \wedge w_2$. However, we do know that $x \leq w_1 \wedge w_2$, and $\mathrm{inv}(x')$ differs from $\mathrm{inv}(x)$ only by the addition of t; thus we specifically have $t \notin \mathrm{inv}(w_1 \wedge w_2)$.

On the other hand, we know that $x' \leq w_1 \vee w_2$. Thus $t \in \mathrm{inv}(w_1 \vee w_2)$. Between this and the previous paragraph, we know $(w_1 \wedge w_2)D$ and $(w_1 \vee w_2)D$ lie on opposite sides of $\mathrm{Fix}(t)$. This is very limiting: it implies that $\mathrm{Fix}(t)$ must be one of the hyperplanes appearing in (or implied by the ellipses in) the figure above.

Further, since $x \leq w_1 \wedge w_2$, xD lies on the same side of $\mathrm{Fix}(t_1)$ and $\mathrm{Fix}(t_2)$ as $(w_1 \wedge w_2)D$. In particular, it cannot have any of the hyperplanes in the diagram above as a border except $\mathrm{Fix}(t_1)$ and $\mathrm{Fix}(t_2)$. Putting this together with the above, we then know $\mathrm{Fix}(t)$ is one of these two hyperplanes; without loss of generality, $t = t_1$.

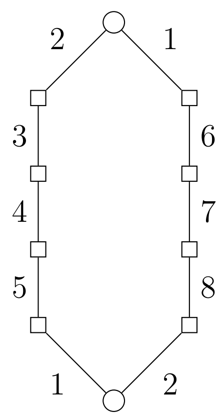
Finally, knowing $t = t_1$ gives us that

$$\mathrm{inv}(x') = \mathrm{inv}(x) \cup \{t_1\} \subset \mathrm{inv}(w_1)$$

but this implies $x' \leq w_1$ in weak order. Since $x' \leq y$ as well, we have $x' \leq y \wedge w_1 = x$, a contradiction. $\square$

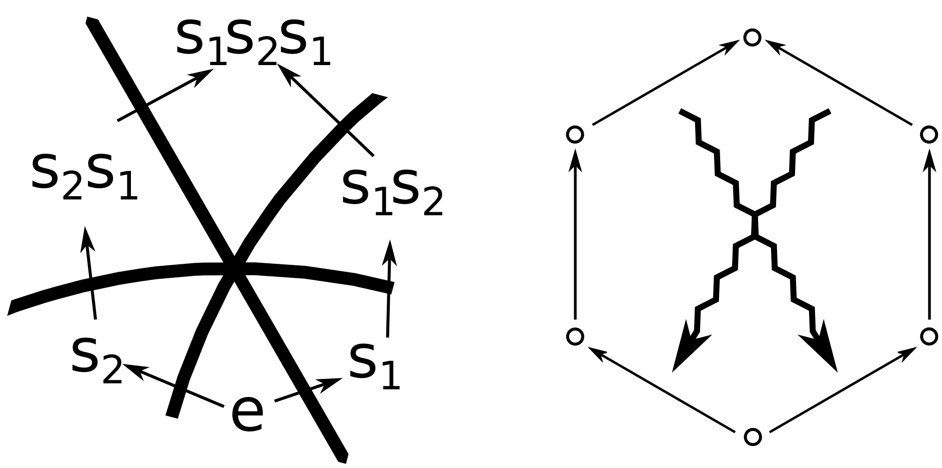
Now let's return to looking at a couple of examples of weak order, paying particular attention to the join-irreducible elements, slide equivalences, and forcing relations.

- Consider type $I_2(5)$. We label each slide equivalence class of covers by a number. Join-irreducible elements are marked by squares.

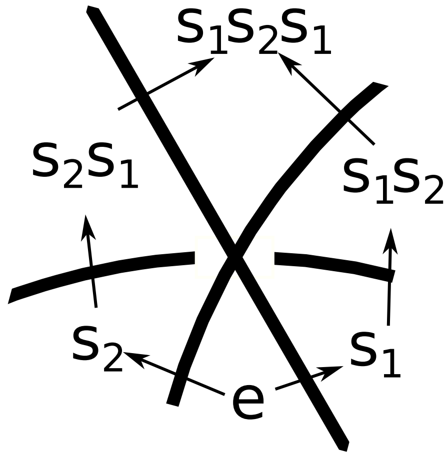


Among the slide equivalence relations, the forcing relations are that 1 and 2 force everything else.

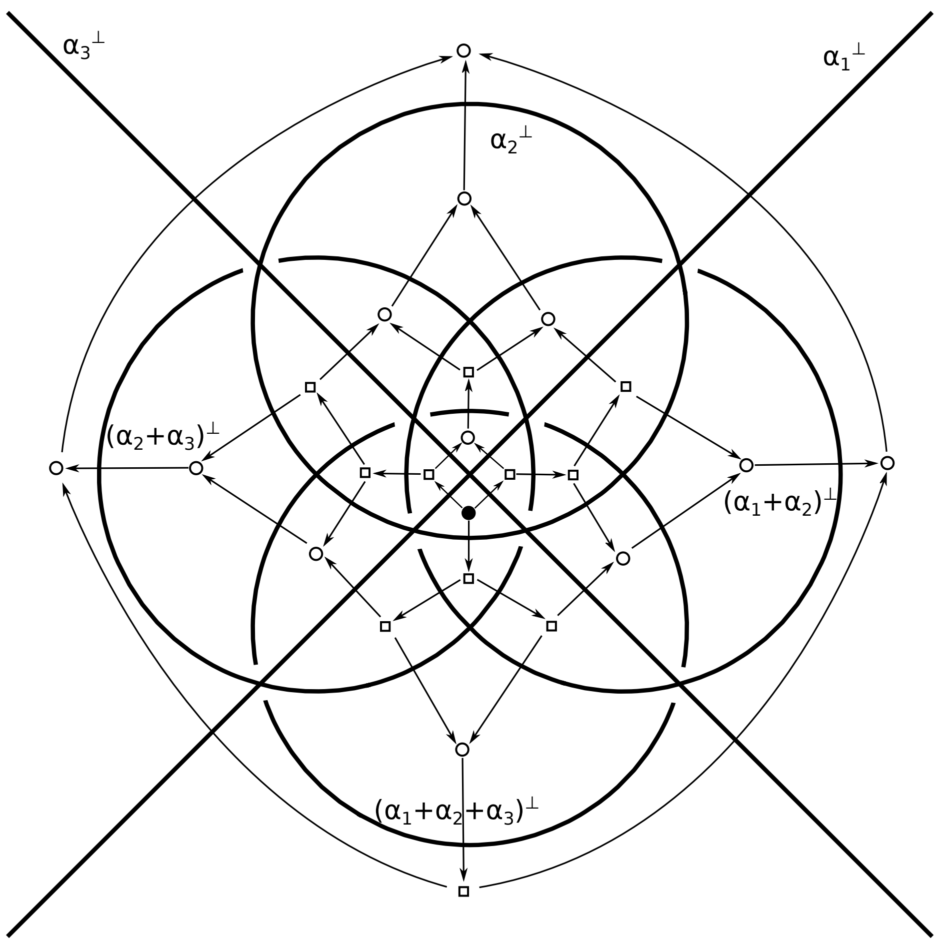
- Consider type A_3. As before, we consider its hyperplane arrangement in stereographic projection. At every place where 3 planes meet, the Hasse diagram of the weak order will contain a hexagon, shown on the left in the figure below. Within this hexagon there will be two nontrivial slide equivalences, shown on the right.



Looking at cover relations as segments of the hyperplane arrangement, we can capture the fact that the two middle covers are *not* slide equivalent by breaking the hyperplane they both lie on:



We illustrate the full breakdown of the A_3 arrangement in this way. As before, we mark the join-irreducible elements by squares:



We now segue into discussing one of the homework problems from last week, on counting the number of join-irreducible elements in S_n, by looking at what join-irreducibility in S_n means concretely in terms of permutations. Recall that a join irreducible element in right weak order is an element with a unique right descent s_k. Looking at permutations just as rearranged sequences of numbers, for the permutation $a_1 a_2 \cdots a_n$ to have unique right descent s_k is the same as saying that

$$a_1 < a_2 < \cdots < a_k > a_{k+1} < a_{k+2} < \cdots < a_n$$

(which clarifies the use of the term "descent"). There are a couple of ways of counting join-irreducibles.

First, we can fix a particular descent k and count how many join-irreducibles have that descent. We can construct a permutation satisfying the inequalities above by choosing almost any k elements and arranging them in order to make $a_1 < \cdots < a_k$, while similarly placing the remaining elements in order to make $a_{k+1} < \cdots < a_n$. The only context in which this fails is if we choose $a_1, \ldots, a_k$ to be $1, \ldots, k$, since then a_{k+1} cannot be less than a_k. Since every other choice works, this gives us $\binom{n}{k} - 1$ join-irreducibles with unique descent s_k. The total number of join-irreducibles is then

$$\sum_{k=0}^{n} \left(\binom{n}{k} - 1 \right) = 2^n - (n+1).$$

We can also keep track of the pair of descending values (a_k, a_{k+1}) rather than the index k. This pair of values tells us which hyperplane induces the cover associated to our join-irreducible element. So how many permutations have left descent (j, i), with $j > i$? Such a permutation is determined by, for every $\ell \neq j, i$, a choice of whether it falls before or after the pair j, i in our permutation. However, the inequalities we started with impose some limitations: if $\ell > j$, we cannot place it on the left side of j, and if $\ell < i$, we cannot place it on the right side of i. Thus a permutation with unique right descent given by the pair $j > i$ is defined by a choice of left or right for each ℓ between i and j. This gives 2^{j-i-1} different options.

Looking at the above figure, in which we broke the A_3 hyperplane arrangement into pieces associated to slide equivalence classes, this explains why the number of pieces each plane broke into was a power of 2. Because the weak order is semidistributive, there should be exactly one join-irreducible in each slide equivalence class, so the number of pieces should be the same as the number of join-irreducibles whose associated cover is induced by that plane. We just calculated that to be a power of 2.

For example, consider the hyperplane $\mathrm{Fix}(1\ 3) = (\alpha_1 + \alpha_2)^\perp$. The join-irreducibles separated from their predecessors by this hyperplane are those whose unique descent is given by 31, and following through our above reasoning such a join-irreducible is determined by where we place 2, to get either 2314 or 3124. This corresponds to the plane $(\alpha_1 + \alpha_3)^\perp$ being broken into two pieces.

Next time, we'll look at forcing relations in Coxeter groups systematically. After that, we'll consider some special quotients of weak order, and move on to strong order with whatever time remains in the semester.

November 13 - Shards and Slide Equivalence

For a Coxeter Group W, we know that for any $a \in W$, $L = [e, a]$ defines a lattice. The goal of today is to understand and characterize slide equivalence classes of Coxeter Groups.

In all of our pictures, we have seen that, if (x_1, y_1) and (x_2, y_2) are slide equivalent, then the same hyperplane separates $x_1 D$ from $y_1 D$ and separates $x_2 D$ from $y_2 D$. We now prove this in general.

Proposition. Let W be a Coxeter Group. Choose $a \in W$, and let $L = [e, a]$. Choose $(x_1, y_1), (x_2, y_2) \in \text{Covers}(L)$, such that $(x_1, y_1) \rightsquigarrow (x_2, y_2)$. Let $y_1 = t_1 x_1$ and $y_2 = t_2 x_2$. Then, $t_1 = t_2$.

Proof. We have $\text{inv}(y_j) = \text{inv}(x_j) \cup \{t_j\}$. Since $x_1 \geq x_2$ but $x_1 \not\geq y_2$, we must have $t_2 \notin \text{inv}(x_1)$. But then, since $y_1 \geq y_2$, we must have $t_2 \in \text{inv}(x_1) \cup \{t_1\}$. So $t_2 = t_1$, as desired. □

We now work towards our goal of characterizing slide equivalence classes of a Coxeter Group. Nathan Reading's theory of ***shards*** is meant to describe the slide equivalence classes for some fixed reflection $t \in T$. This theory was originally laid out in Nathan Reading, "Lattice congruences of the weak order", (2004), but a much more readable source (also by Reading) is Chapters 9 and 10 of *Lattice Theory: Special Topics and Applications, Volume 2*. I have drawn heavily on the latter source in preparing these lectures.

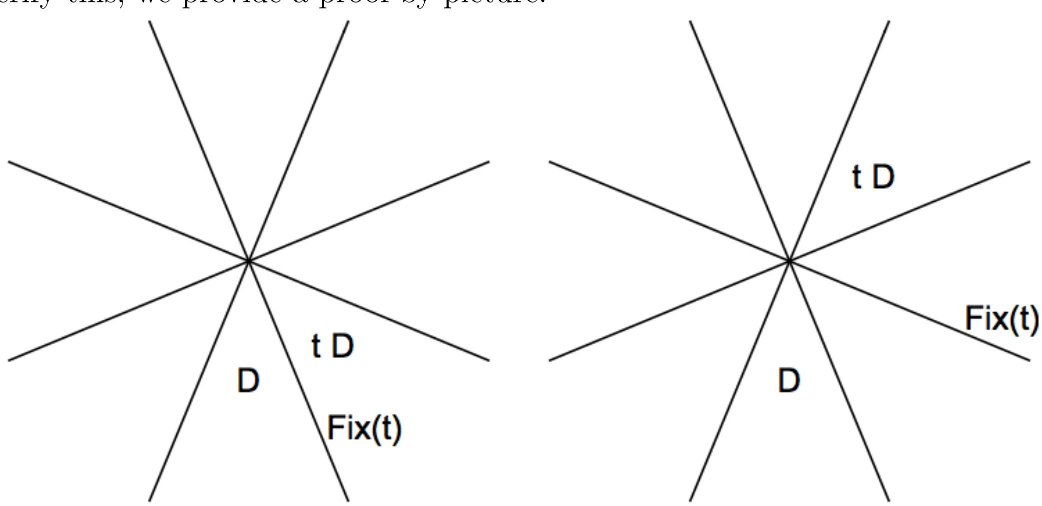
Fix a reflection representation V of W with corresponding $D_0 \neq 0$. We then consider a face inside the interior of the Tits Group with codimension 2. Such a face is stabilized by a rank 2 subgroup, while the interior is stabilized by a finite group. The neighborhood of the face looks like an $I_2(h)$ arrangement, and there is a canonical bottom region, which contains D. We will call the hyperplanes bounding this bottom region the ***fundamental hyperplanes***.

For any two roots β and γ, we say that $\beta^\perp$ ***cuts*** $\gamma^\perp$ if $\beta^\perp \cap \gamma^\perp \cap (\text{Tits})^\circ \neq \emptyset$, and, in the parabolic containing β and γ, the hyperplane $\beta^\perp$ is fundamental and $\gamma^\perp$ is not. In this case, we call $\beta^\perp \cap \gamma^\perp$ a ***fracture*** of $\gamma^\perp$.

Lemma. Let $t \in T$. The fractures of $\text{Fix}(t)$ are precisely

$$\{\text{Fix}(t) \cap \text{Fix}(u) \mid \text{Fix}(t) \cap \text{Fix}(u) \cap \text{Tits}^\circ \neq \emptyset, u \in \text{inv}(t), t \neq u\}$$

Proof. Let F be a codimension 2 subspace of the Coxeter arrangement, chosen such that $F \subset \text{Fix}(t)$, and $F \cap \text{Tits}^\circ \neq \emptyset$. This implies that $\text{Stab}(F)$ is a rank 2 parabolic, denoted P. It is sufficient to show that $\text{Fix}(t)$ is fundamental if and only if $\nexists u \in \text{inv}(t) \cap P$ that is not t. To verify this, we provide a proof by picture:



The left case is where $\text{Fix}(t)$ is fundamental, and the right case is where $\text{Fix}(t)$ is not. In the left case, it follows that reflecting over t does put D above any hyperplane of the arrangement other than $\text{Fix}(t)$, whereas, in the right case, reflecting over t places D above other hyperplanes as well as $\text{Fix}(t)$. □

This result immediately yields the following useful corollary:

Corollary. Fix(t) has finitely many fractures.

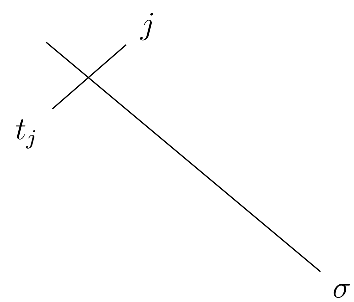
The fractures form a hyperplane arrangement for Fix(t), and the reflections in this arrangement with nontrivial intersection with Tits° are called ***Shards***. We will write Ш(t) to denote the set of shards of Fix(t), and Ш or Ш(W) for the shards of $\bigsqcup_{t\in T}$ Fix(t).

Theorem. Ш is in bijection with the slide equivalence classes of W. In particular, $(x_1, y_1) \rightsquigarrow (x_2, y_2)$ if and only if they border along the same shard.

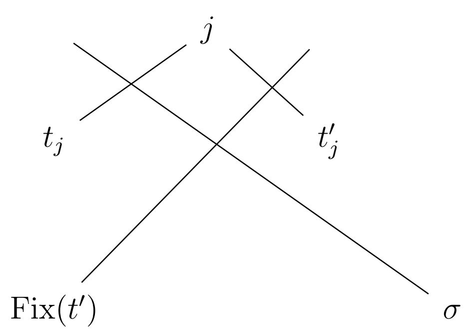
Proof. First, suppose that (x_1, y_1) and (x_2, y_2) border along the same shard of Fix(t). We choose points p_1 and p_2 generically from $x_1D \cap y_1D$ and $x_2D \cap y_2D$ respectively, and look at the line segment $\overline{p_1p_2}$. A path from $p_1 \to p_2$ crosses through subspaces which are of codimension 2. This yields a sequence of covers: (x^1, y^1), (x^2, y^2) $\ldots (x^N, y^N) = (x_2, y_2)$.

Shards are convex, implying that this path does not cross any hyperplane that cuts Fix(t). In other words, this crosses no fractures of Fix(t). Thus, in every crossing, Fix(t) is fundamental. Therefore, in every crossing, either $(x^j, y^j) \rightsquigarrow (x^{j+1}, y^{j+1})$ or vice versa. We have verified that the if two points border along the same shard, they are slide equivalent.

We now verify the converse case. It is enough to show that every shard borders a join irreducible element. For a shard σ, Let $j \in W$ be a minimal element above σ.



Where j covers t_j. Suppose that j also covers t_j. Then we have



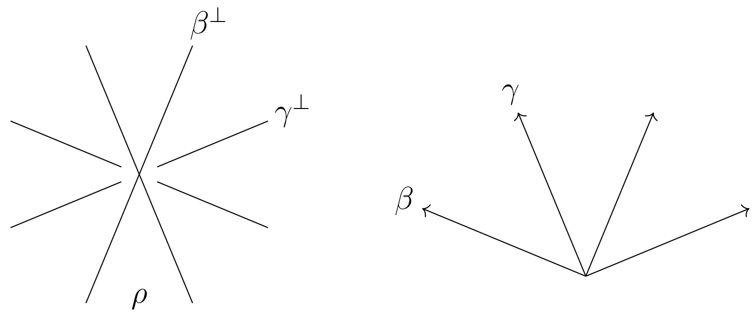
This implies that Fix(t) is fundamental in Fix(t) $\cap$ Fix(t'), implying that t_j borders σ, a contradiction to minimality. $\square$

November 15 – Congruence Uniformity, part 1

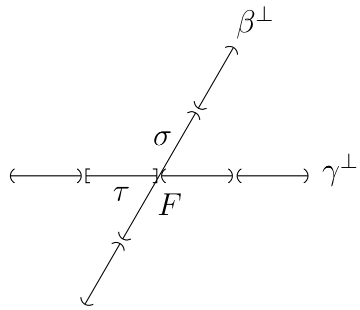
Our next big goal is to show that (intervals in) Coxeter groups are congruence uniform, which means that the forcing relation on slide equivalence classes is acyclic. We now know that slide equivalence classes are shards, but we don't have a good understanding of forcing.

As partial progress towards this goal, we will show that forcing is the transitive closure of a more immediate relation we call cutting. To be honest, I still don't feel that I have a good understanding of forcing, but this will get us enough understanding to be able to prove the CU property.

Recall that we said $\beta^\perp$ ***cuts*** $\gamma^\perp$ if β and γ lie in a finite rank two parabolic (equivalently, $\beta^\perp \cap \gamma^\perp \cap \mathrm{Tits}^\circ = \emptyset$), with β fundamental and γ not fundamental:



We called $\beta^\perp \cap \gamma^\perp$ a ***fracture*** of $\gamma^\perp$. Let $\beta^\perp$ cut $\gamma^\perp$ at fracture F, let σ be a shard of $\beta^\perp$, and let τ be a shard of $\gamma^\perp$. We say σ ***cuts*** τ if $(\tau \cap F)^\circ \subset \sigma^\circ$ and $\tau \cap F$ is a codimension 1 face of the cone τ.



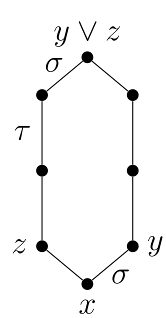
Forcing is the transitive closure of cutting, as a preorder on Ш. Moreover, cutting is an acyclic relation on Ш. Temporarily, let $\longrightarrow_c$ denote the transitive closure of cutting, e.g. $\sigma \longrightarrow_c \tau$.

Choose x generating $(\tau \cap F)^\circ$, so that a neighborhood of x looks like a rank two interval in the Coxeter group, with σ fundamental and τ non-fundamental.

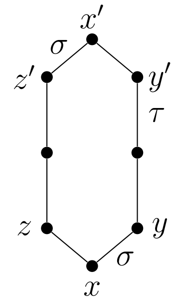
We claim that if σ cuts τ, then σ forces τ.

To see this, let x be the bottom element of the interval, let $y \gtrdot x$ be labelled by σ, and let z be the other cover of x. We have two cases:

(1) If τ is in the chain above z, then τ labels a cover in a chain from $x \vee z = z$ to $y \vee z$, so $\sigma \Rightarrow \tau$.



(2) Suppose instead that τ is in the chain above y. Let $y \leq y' \lessdot y \vee z$, $z \leq z' \lessdot y \vee z$, with τ labeling a cover in $[y, y']$. Then σ labels $z' \lessdot y' \vee z =: x'$ and τ labels an edge in a chain from $z' \wedge y'$ to $x' \wedge y'$.



We now wish to prove the converse, namely that $\Longrightarrow$ implies $\longrightarrow_c$.

Lemma. Let $u \leq v$ in W, and take two chains

$$u = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_N = v,$$
$$u = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_N = v.$$

Then the $\longrightarrow_c$ closure of $\{(x_j, x_{j+1})\}$ is the same as of $\{(y_j, y_{j+1})\}$.

Proof. By a lemma from homework, it is enough to consider the case when the path differ by a single braid move, i.e. the paths correspond to opposite sides of a rank two interval. The first and last cover of the interval cut or equal all the shards on the other side of the interval. □

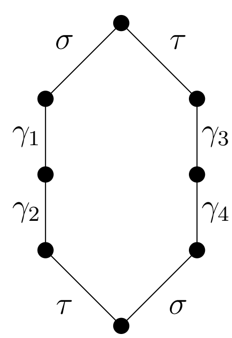
The rest of the proof was badly done, so we stop the notes here.

November 18 – Congruence Uniformity, part 2

Recall from last time: We tried to show that forcing is the transitive closure of cutting. Let's review some background and go through a slightly different presentation.

Let $L = [e, a] \subseteq W$ be a finite interval of a Coxeter group. We have two relations on Covers(L):

- $\Rightarrow$ denotes the transitive closure of forcing; recall that (x, y) forces (x', y') if
  - $x \vee z \leq x' \lessdot y' \leq y \vee z$; or
  - $x \wedge z \leq x' \lessdot y' \leq y \wedge z$
- (A temporary notation) $\rightarrow_C$ is a relation on shards/slide equivalence classes which we define as the transitive closure of the relation on shards/slide equivalence classes in rank 2 intervals as follows:



We say that σ, τ cut the γ_i.

If σ cuts τ, then σ also forces τ, and this doesn't change when we take transitive closure. What we want to show is that we can get from σ to τ via a sequence of cutting relations.

Recall the following lemma from last class:

Lemma. Let $u \leq v$ and let

$$u = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_N = v$$

$$u = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_N = v$$

Then the set of shards in $\to_C$ of $\{(x_j, x_{j+1})\}$ is the same as the set of shards in $\to_C$ of $\{(y_j, y_{j+1})\}$.

Note that the chains necessarily have the same length because we are working with Coxeter groups which are graded by the length function.

Definition. For $u \leq v$, and take any chain $u = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_N = v$. Define

$$\mathcal{C}(u, v) = \{(x', y') \in \text{Covers}(L) : \exists_j (x_j, x_{j+1}) \to_C (x', y')\}$$

Our lemma then shows that $\mathcal{C}(u, v)$ is independent of the choice of chain.

In terms of $\mathcal{C}(u, v)$, what we want to show is that for $x \lessdot y$.

$$\mathcal{C}(x, y) \supseteq \mathcal{C}(x \vee z, y \vee z)$$

$$\mathcal{C}(x, y) \supseteq \mathcal{C}(x \wedge z, y \wedge z)$$

In fact, we will show this for $x \leq y$, not just $x \lessdot y$

We can eliminate z from our notation as follows: let $x' = x \vee z$, $y' = y \vee z$. Then $y \vee z = y \vee (x \vee z) = y \vee z'$. So if $x \leq x'$, what we need to show is:

(*) $$\mathcal{C}(x, y) \supseteq \mathcal{C}(x', x' \vee y)$$

and if $y' \leq y$, then

$$\mathcal{C}(x, y) \supseteq \mathcal{C}(x \wedge y', y')$$

Proof of $(*)$. We prove $(*)$ by induction on $\ell(a) - \ell(x)$. The proof for the meet statement is similar.

Base Case: $x = a = x' \vee y = y$, so $\mathcal{C}(x, y) = \mathcal{C}(x', x' \vee y)$.

Inductive Step: Choose a chain $x = x_1 \lessdot x_2 \leq x'$. Since x_2 has length greater than x, $l(a) - l(x_2) < l(a) - l(x)$, so the inductive hypothesis tells us that $\mathcal{C}(x_2, x_2 \vee y) \supseteq \mathcal{C}(x', x' \vee y)$.

We need to show that $\mathcal{C}(x, y) \supseteq \mathcal{C}(x_2, x_2 \vee y)$. We do this in two steps. First, choose $x \lessdot x_3 \leq y$, and sketch the diagram with the rank 2 parabolic generated by x_2, x_3.

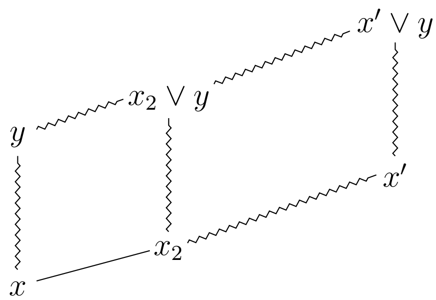
Step 1: We "move" the chain $x_3 \rightsquigarrow y$ to a chain $x_2 \vee x_3 \rightsquigarrow x_2 \vee y$. That is, $x_3 > x$, so $l(a) - l(x_3) < l(a) - l(x)$, so the inductive hypothesis tells us that $\mathcal{C}(x_3, y) \supseteq \mathcal{C}(x_2 \vee x_3, x_2 \vee y)$.

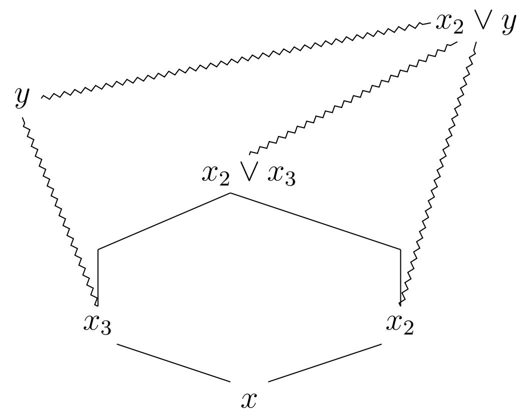
Step 2: Now in the rank 2 parabolic, we know that every cover between x_2 and $x_2 \vee x_3$ is cut by or slide equivalent to (x, x_3).

It follows that $\mathcal{C}(x, y) \supseteq \mathcal{C}(x_2, x_2 \vee y)$, since the covers in some chain from x to y force every edge in a chain from x_2 to $x_2 \vee y$. Thus, $(*)$ is proved. □

Now we are ready to prove congruence uniformity! Recall that congruence uniformity means that forcing preorder on slide equivalence classes is a partial order. We now know that slide equivalence classes are shards, and forcing is the same as cutting.

We want to show that the directed graph of cutting has no oriented cycles. The most natural way to do this is to put a total order on the set of shards so that if σ cuts τ, then σ





comes before τ. To do this, we will put a total order on hyperplanes: specifically, if $\beta^\perp$ cuts $\gamma^\perp$, then $\beta < \gamma$. Between shards in the same hyperplane, we break ties arbitrarily (since none of them will cut each other).

Choose a *symmetric* Cartan matrix, i.e. choose the roots to have length $\sqrt{2}$. Choose the α_i to be linearly independent. Let $\rho \in V^\vee$ with $\langle \rho, \alpha_i \rangle = 1$. Order Φ^+ according to $\langle \rho, \ \rangle$, breaking ties arbitrarily. We are left to show that if $\beta^\perp$ cuts $\gamma^\perp$, then $\langle \rho, \beta \rangle < \langle \beta, \gamma \rangle$.

Look at a finite rank 2 subsystem with positive roots $\beta_1, \beta_2, \ldots, \beta_m$ in order, so that β_1, β_m are fundamental.

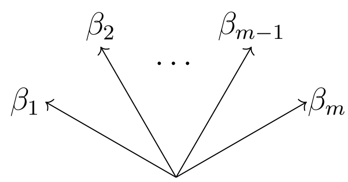


Figure 23. The roots of a rank 2 parabolic subgroup. Note that the fundamental roots (i.e. the outermost roots) are labelled β_1, β_m.

Exercise.

$$\beta_k = \frac{\sin \frac{\pi k}{m}}{\sin \frac{\pi}{m}} \beta_1 + \frac{\sin \frac{p(m+1-k)}{m}}{\sin \frac{\pi}{m}} \beta_m$$

Pairing this all with ρ, we get

$$\langle \rho, \beta_k \rangle = \frac{\sin \frac{\pi k}{m}}{\sin \frac{\pi}{m}} \langle \rho, \beta_1 \rangle + \frac{\sin \frac{p(m+1-k)}{m}}{\sin \frac{\pi}{m}} \langle \rho, \beta_m \rangle \geq \langle \rho, \beta_1 \rangle + \langle \rho, \beta_m \rangle$$

With the last inequality because: (1) the coefficients are always ≥ 1; and (2) since the β_i are positive roots, ρ pairs positively with β_1, β_m. It follows that $\langle \rho, \beta_k \rangle \geq \langle \rho, \beta_1 \rangle, \langle \rho, \beta_m \rangle$

November 20 - Shards and Forcing in Type A

Recall the type A representation. We will look at some quotients of this lattice. From last class, to do this, we just need to understand how cutting behaves on the lattice.

Our convention is that the positive roots are $\{e_j - e_i\}_{1 \leq i < j \leq n}$. The rank two parabolic subgroups are either of the form $\langle (ij), (kl) \rangle \cong A_1 \times A_1$ (with i, j, k, l distinct) or $\langle (ij), (jk) \rangle \cong$

A_2 (with i, j, k distinct. The corresponding positive roots are $\{e_j - e_i, e_l - e_k\}$ (with $i < j, k < l$) and $\{e_k - e_j, e_j - e_i, e_k - e_i\}$ (with $i < j < k$).

In $A_1 \times A_1$ sub-systems, there is no cutting. In an A_2-subsystem, with $i < j < k$, the hyperplanes $(e_j - e_i)^\perp$ and $(e_k - e_j)^\perp$ cut $(e_k - e_i)^\perp$. The two pieces that $(e_k - e_i) \perp$ are cut into are $\{x | x_j < x_i = x_k\}$ and $\{x | x_j > x_i = x_k\}$. We see this in figure 1.

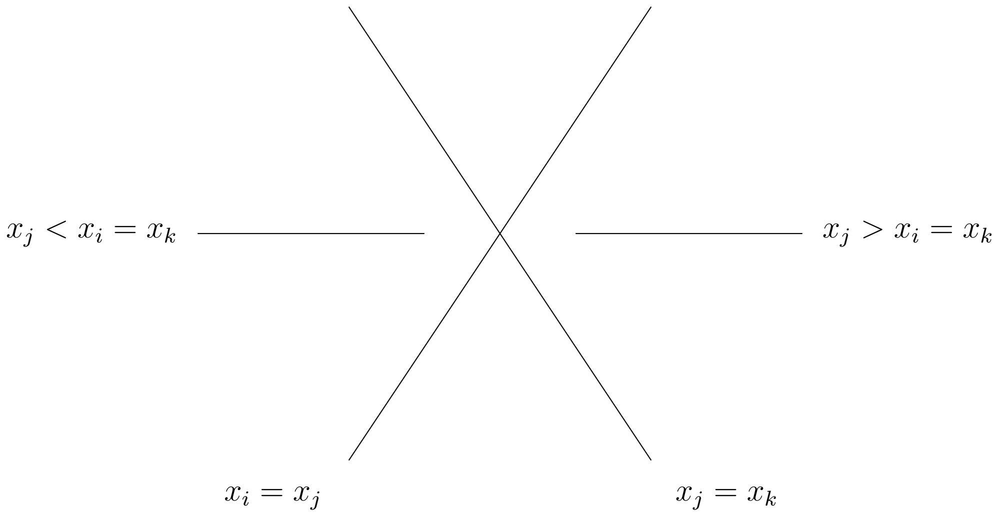


FIGURE 24. Cutting of the hyperplanes for the parabolic subgroup $\langle (ij), (jk) \rangle$

In general, the hyperplane $\{x_i = x_k\}$ (with $i < k$) will be cut into 2^{k-i-1} pieces, as for each j in the open interval (i, k) we have the choice to have $x_j > x_i = x_k$ or $x_j < x_i = x_k$.

Definition. Suppose $(i, k) = L \sqcup R$ is a partition. We define $\sigma(L|_i^k|R)$ to be the shard $\{x | x_j < x_i = x_k$ for $j \in L$, and $x_j > x_i = x_k$ for $j \in R\}$.

We now draw Figure 1 using this notation, and focus on the shards rather than the hyperplanes in Figure 2.

With this, because forcing is the transitive closure of cutting, we have that the shard $\sigma(L_1|_{i_1}^{k_1}|R_1)$ forces the shard $\sigma(L_2|_{i_2}^{k_2}|R_2)$ if and only if $i_2 \leq i_1 < k_1 \leq k_2$ and $L_1 = L_2 \cap (i_1, k_1)$ and $R_1 = R_2 \cap (i_2, k_2)$.

Now, in order to describe some of the quotients, we just need to pick some shards to collapse, E, making sure that E is closed under the above relation. In actuality, it is easier to describe the complement K of E, the shards we do not collapse.

Example. Let $K = \{\sigma(\emptyset|_K^{k+1}|\emptyset)\}$. This corresponds to getting rid of all of the hyperplanes except $(e_{k+1} - e_k)^\perp$. We have seen this quotient before, the quotient is the map we saw on Problem Set 8, question 3, sending an element w to the set of its left descents.

Example. Let W_I be any parabolic subgroup of A_n, and let Φ_I be the corresponding subroot system. Then we can take $K = \{\sigma(L|_I^K|R) | e_k - e_i \in \Phi_I\}$. We have also seen this one before, this is the map $W \to W_I$ that sends an element w to w_I.

Example. The previous two examples have a common generalization: Let $J \subseteq \Phi^+$ be a set so that if $\beta^\perp$ cuts $\gamma^\perp$ and $\gamma \in J$, then $\beta \in J$. Then we take $K = \{\sigma(L|_i^k|R) | e_k - e_i \in J\}$.

Example. Let $K = \{\sigma((i, k)|_i^k|\emptyset)\}$. Figure 3 shows what this looks like in the case of A_2.

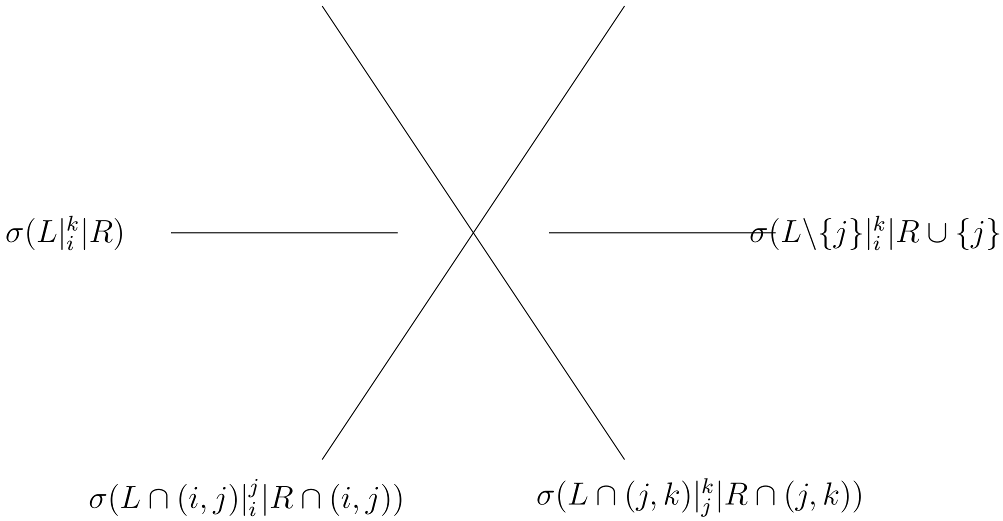


FIGURE 25. The different shards corresponding to the parabolic subgroup $\langle (ij),(jk)\rangle$

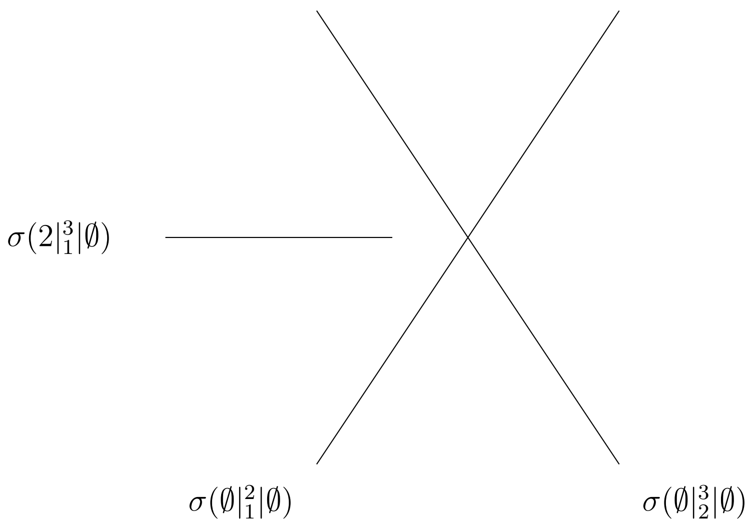


FIGURE 26. What the quotient of Example 4 looks like on A_2

If this looks familiar, that is because this is the same as the Tamari lattice T_3, and in fact quietening out by K in general will give the Tamari lattice T_n.

To show this, we will show that what we have constructed matches up with Loday's construction (Problem Set 7, question 3). In the quotient we have just constructed, we have that a shard $\sigma(L|_i^k|R)$ is collapsed if and only if $R \neq \emptyset$ if and only if there is some $j \in (i,k)$ with $x_j > x_i = x_k$. And to show that the edge gets collapsed in Loday's construction, we need to show that the points

$$(\ldots, x_i, \ldots, x_j, \ldots, x_k, \ldots)$$
$$(\ldots, x_k, \ldots, x_j, \ldots, x_i, \ldots)$$

have the same tree.

Well, if such a j exists, we have the point (j, x_j) above the points (i, x_i) and (k, x_k) in the trees, so they will be in different sub trees, so there order does not matter.

Likewise, if switching x_i and x_k does not change the tree, then (i, x_i) and (k, x_k) cannot be above one another in the tree, so there must be something between them that is above both of them.

Example. We can generalize the previous construction in the following way: let $[n] = \mathcal{L} \sqcup \mathcal{R}$ be a partition. Define $K = \{\sigma(\mathcal{L} \cap (i,k)|_i^k|\mathcal{R} \cap (i,k)\}$. The previous construction corresponds to $\mathcal{L} = [n]$. Note that it does not matter where $n, 1$ go in the partitions. These are the Cambrian quotients, and we'll see more of them next time.

November 22 – Cambrian quotients in type A

Recall the setup from last time: we partition $[n] = \mathcal{L} \sqcup \mathcal{R}$, and keep only the shards of the form $\sigma\left(\mathcal{L} \cap (i,k) \left|_i^k\right| \mathcal{R} \cap (i,k)\right)$. We'll biject equivalence classes to trivalent planar trees with $|\mathcal{L}| + 1$ roots and $|\mathcal{R}| + 1$ leaves. Here we are using the "real world" convention that roots are at the bottom and leaves are at the top, as opposed to the opposite convention from the computer science literature. Also, Professor Speyer apologizes for defining his notation such that $\mathcal{L}$ corresponds to **r**oots and $\mathcal{R}$ to **l**eaves.

Let $(y_1, \ldots, y_n) \in \mathbb{R}$ with distinct entries. Draw points at (j, y_j), and draw vertical rays down from (j, y_j) if $j \in \mathcal{L}$, up if $j \in \mathcal{R}$. Sufficiently far down on the page there are $|\mathcal{L}| + 1$ regions, and sufficiently far up there are $|\mathcal{R}| + 1$ regions. In each of these $|\mathcal{L}| + 1$ regions at the bottom, start with 1 root and go up. When reaching a vertex, merge or split the branches as appropriate (so at a vertex with a down ray, the two branches will merge, and at a vertex with an up ray, the one branch will split). See Figure 27 for an example. We claim that two such points give different trees if and only if they are separated by one of our shards.

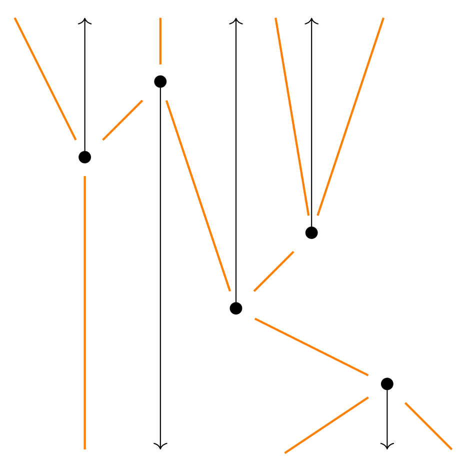


Figure 27. The tree (in orange) generated from $\mathcal{L} = \{2, 5\}$, $\mathcal{R} = \{1, 3, 4\}$.

Let $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ differ only by exchanging elements in positions i and k, and suppose that there is no j with $y_i < y_j < y_k$ (and likewise with z). This means that y and z are related via reflecting over the $i = k$ hyperplane, and that this is the only hyperplane separating them. Suppose that the shard between them, $\sigma\left(L \left|_i^k\right| R\right)$, is *not*

$\sigma\left(\mathcal{L}\cap(i,k)\left|{}_{i}^{k}\right|\mathcal{R}\cap(i,k)\right)$. Then either there is some $j\in\mathcal{L}\cap R$ with $i<j<k$, or there is some $j\in L\cap\mathcal{R}$ with $i<j<k$. The two cases are symmetrical, so we address the first case. Having $j\in R$ is equivalent to saying $y_j>y_i,y_k$, and having $j\in\mathcal{L}$ means the ray gets drawn downwards. So although the vertices for i and k switch relative positions, they are still separated by the ray for j, and so the topology of the tree doesn't change. This is depicted in Figure 28

FIGURE 28. The case when $j\in\mathcal{L}\cap R$. To the left is the ray diagram for y, to the right is that for z.

Conversely, suppose that $L=\mathcal{L}\cap(i,k)$ and $R=\mathcal{R}\cap(i,k)$. We have two cases, based on whether i and k are in the same part of the partition or not. In case 1, if both i and k are in the same part of the partition, say $i,k\in\mathcal{L}$, then their rays point in the same direction, and there are no rays between them. This case is in Figure 29. In Case 2, if i and k are in distinct parts, then their rays point in opposite directions, and there are still no rays separating them. This case is in Figure 30.

FIGURE 29. Case 1, where $i,k\in\mathcal{L}$. To the left is the ray diagram for y, to the right is that for z.

FIGURE 30. Case 2, where $i\in\mathcal{L}$ but $k\in\mathcal{R}$. To the left is the ray diagram for y, to the right is that for z.

This description in terms of trees is due to David Speyer, but is not too hard to see it is equivalent to the description in terms of secondary polytopes in Reading's "Cambrian Lattices", (2006).

Now we look at some other ways to think about Cambrian quotients of S_n.

Lemma. A lattice quotient $q : L \to L/\theta$ of L a finite lattice is determined by the set $C \subset L$ of bottom elements of equivalence classes.

Proof. Let $\pi_\downarrow : L \to C$ send $x \in L$ to its bottom representative. Such an element exists because equivalence classes are closed under meet, since if $q(x_1) = q(x_2)$, then

$$q(x_1 \vee x_2) = q(x_1) \vee q(x_2) = q(x_1) = q(x_2).$$

Now we claim that $\pi_\downarrow(x) = \bigvee_{c \in C,\ c \leq x} c$. We want to show that if $c \in C$ with $c \leq x$, then $c \leq \pi_\downarrow(x)$. So we want to show that $\pi_\downarrow(x) \vee c = c$. We know that $x \vee c = c$, so $q(x) \vee q(c) = c$, i.e. $q(x) = q(\pi_\downarrow(x)) \geq q(c)$. From the worksheet, we can lift this inequality and fix the upper end, so that we get $\pi_\downarrow(x) \geq y$ for some y with $q(y) = q(c)$. Since c is the bottom element of its equivalence class, this means $\pi_\downarrow(x) \geq y \geq c$, as desired. □

What are the bottom elements of Cambrian quotients? They are elements of the form $w = w_1 \cdots w_n$ in one-line notation in S_n, where if descend $w_a = k > w_{a+1} = i$, then for all $j \in (i, k)$, if $j \in \mathcal{L}$, then j can't be right of position a, and if $j \in \mathcal{R}$, then it can't be left. This condition almost looks like pattern avoidance, except that pattern avoidance doesn't normally force two elements to be in adjacent positions.

Lemma. If we have a w of the above form, then consecutivity is not actually required. In other words, if $j \in \mathcal{L}$ and $i < j < k$ then we can't have $\ldots k \ldots i \ldots j \ldots$; if $j \in cR$ and $i < j < k$ then we can't have $\ldots j \ldots k \ldots i \ldots$.

Proof. The cases are symmetric so we only address the case of $i < j < k$ and $j \in \mathcal{L}$. Suppose we have such an (i, j, k) triple where i and k are as close as possible. By assumption, they can't be adjacent so let h be between k and i. Now we have two cases:

(1) If $h < j$, then we have $h < j < k$, $j \in \mathcal{L}$, appearing in the order $\ldots k \ldots h \ldots j \ldots$ within w. But k and h are closer than k and i, a contradiction.
(2) $j < h$, then $i < j < h$, $j \in \mathcal{L}$, appearing in the order $\ldots h \ldots i \ldots j \ldots$ within w. But h and i are closer than k and i, a contradiction.

□

So for example if $\mathcal{R} = \emptyset$, then this means bottom elements are those avoiding the permutation 312.

November 25 – Cambrian quotients in other types

This lecture covers ideas started by Nathan Reading in "Cambrian Lattices" (2006), "Clusters, Coxeter-sortable elements and noncrossing partitions" (2007) and "Sortable elements and Cambrian lattices" (2007) and then further developed by Reading and David Speyer in "Sortable elements in infinite Coxeter groups" (2008) and "Sortable Elements for Quivers with Cycles" (2010). The earlier papers are all in finite type and rely on a lot of computer checks, the papers with Speyer are uniform. The first Reading-Speyer paper has an acyclicity hypothesis which we figured out how to remove in the second paper.

I had hoped that, after giving all these lectures and understanding lattice congruences of weak order so much better, these proofs would become so much slicker. They didn't. So this lecture is going to be a survey of results without proofs, because the proofs are still difficult.

Let W be a Coxeter group, let Γ be its Coxeter diagram and let Ω be an orientation of the graph Γ. For example, if W is S_n then Γ looks like

$$(12) \text{ —— } (23) \text{ —— } (34) \text{ —— } \cdots \text{ —— } (n-1\ n)$$

We orient $(j-1\ j) \leftarrow (j\ j+1)$ if $j \in \mathcal{L}$ and $(j-1\ j) \rightarrow (j\ j+1)$ if $j \in \mathcal{R}$.

Our goal is to describe quotients $W \twoheadrightarrow \mathrm{Camb}_\Omega$, which are lattice quotients on each interval $[e,a]$. We will give four such descriptions. Unfortunately, it is not clear that any of them are equivalent to each other, or that any of them are lattice quotients! We first introduce some preliminary notations.

Choose a reflection representation where the α_i are linearly independent. Define a skew symmetric bilinear form ω on V where

$$\omega(\alpha_i,\alpha_j) = \begin{cases} -A_{ij} > 0 & s_i \leftarrow s_j \\ \ \ A_{ij} < 0 & s_i \rightarrow s_j \\ \ \ 0 & s_i s_j = s_j s_i \end{cases}.$$

We can also consider ω as defining a map η from V to $V^\vee$ such that $\omega(\beta,\gamma) = \langle \eta(\beta),\gamma\rangle$. The condition that ω is skew-symmetric corresponds to $\eta(\beta) \in \beta^\perp$.

Description 1 of the Cambrian Quotient A shard σ is **not** collapsed in Camb_Ω if and only if $\eta(\beta)$ is in the relative interior of σ.

Let's see what this means in type A. Which of the shards of dimension $e_k - e_i$ does $\eta(e_k - e_i)$ lie in? Let $i < j < k$. We want to know on which side of the hyperplane $(e_j - e_i)^\perp$ the point $\eta(e_k - e_i)$ lies. In other words, we want to know the sign of $\langle \eta(e_k - e_i), e_j - e_i\rangle = \omega(e_k - e_i, e_j - e_i)$. We have $\omega(e_k - e_i, e_j - e_i) = \omega(e_k - e_j, e_j - e_i) = \omega(\alpha_{k-1} + \cdots + \alpha_{j+1} + \alpha_j,\ \alpha_{j-1} + \cdots + \alpha_{i+1} + \alpha_i)$. If we expand the last expression into terms of the form $\omega(\alpha_q,\alpha_p)$, there is only one nonzero term, $\omega(\alpha_j,\alpha_{j-1})$. This last term is 1 if $j \in \mathcal{R}$ and -1 if $j \in \mathcal{L}$. So, putting $\gamma = \eta(e_k - e_i)$, we have shown that $\gamma_i = \gamma_k < \gamma_j$ if $j \in \mathcal{R}$ and $\gamma_j < \gamma_i = \gamma_k$ if $j \in \mathcal{L}$, matching our previous description.

Unfortunately, it seems rather unclear how to show that this construction is closed under cutting in types other than A, and particularly in infinite types.

Description 2 of the Cambrian Quotient We observed last time that a lattice congruence is determined by the set of minimal elements in the congruence classes. We will call these elements Ω-sortable, and denote the set of them by C.

Here is our second description: Let $v \in W$. Let R be any rank two parabolic of W with roots $\beta_1, \beta_2, \ldots, \beta_m$ ordered so that $\omega(\beta_i,\beta_j) > 0$ for $i < j$.

Roughly, v will be in C if and only if , for any such R, the intersection $\mathrm{inv}(v) \cap R$ is either an initial segment of R, or else the singleton $\{\beta_m\}$. More precisely, m could be infinity and that is fine. Also, it might happen that ω is 0 on R. This happens only in types $A_1 \times A_1$ and $\tilde{A}_1$. In the first case, we impose no restriction on $\mathrm{inv}(v) \cap R$ and, in the second, we insist that $\mathrm{inv}(v) \cap R$ is one of $\emptyset$, $\{\beta_1\}$ or $\{\beta_\infty\}$.

In type A, we are saying that, if $i < j < k$ with $j \in \mathcal{L}$, then we **cannot** have $\mathrm{inv}(v) \cap \{e_j - e_i, e_k - e_i, e_k - e_j\} = \{e_k - e_i, e_k - e_j\}$. In other words, we are avoiding $\cdots k \cdots i \cdots j \cdots$ in this case, as discussed before. The case where $j \in \mathcal{R}$ is similar.

In general, if v obeys the above condition, then it is easy to see that all the shards corresponding to covers (u, v) are not collapsed. However, it doesn't seem obvious how to show the converse. This is the switch between avoiding $\cdots ki \cdots j \cdots$ and avoiding $\cdots k \ldots i \cdots j \cdots$ in type A.

We now introduce another piece of notation. Say that a set J of vertices of Γ is ***Ω-acyclic*** if the orientation Ω restricts to an acyclic orientation on J. If so, let c_J be $\prod_{j \in J} s_j$ ordered with s_i before s_j if Ω orients $s_i \leftarrow s_j$.

Description 3 of the Cambrian Quotient Again, we describe the Ω-sortable elements. They are the elements which have reduced factorizations of the form $c_{J_1} c_{J_2} \ldots c_{J_r}$ where the J_a are Ω-acyclic and $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_r$. It is relatively straightforward to show that any element which has such a factorization obeys the condition in Description 2. Again, the converse seems very unclear.

Let's clarify what these words look like if Ω is acyclic. Let s be a sink of Ω and let Ω' be the orientation where we reverse all edges incident on s to make it a source. We abbreviate $S \setminus \{s\}$ to $\langle s \rangle$.

Suppose that v has a reduced factorization as above. Note that, if $s \in J_1$ then $v \geq c_{J_1} \geq s$ and, if $s \notin J_1$, then v is in the parabolic subgroup $W_{\langle s \rangle}$. More precisely, we claim:

• Let $w \in W$ with $w \geq s$. Then w is Ω-sortable if and only if sw is Ω' sortable. To see this, note that we can transform factorizations $c_{J_1} c_{J_2} \ldots c_{J_r}$ for w into factorizations $c_{J'_1} c_{J'_2} \ldots c_{J'_r}$ for Ω' where $s \in J'_p$ if and only if $s \in J_{p+1}$ and the J'_a are otherwise equal to the J_a.

• Let $w \in W$ with $w \not\geq s$. Then w is Ω-sortable if and only if $w \in W_{\langle s \rangle}$ and is $\Omega|_{\langle s \rangle}$-sortable.

From this, it is not bad to show that Description 3 is equivalent to **Description 4 of the Cambrian Quotient**. For simplicity, take Ω acyclic and let s and Ω' be as above. Then we have $u \equiv v$ in Camb_Ω if and only if one of the following holds:

• $u \geq s$ and $v \geq s$, and $su \equiv sv$ in $\mathrm{Camb}_{\Omega'}$.

• u and $v \in W_{\langle s \rangle}$ and $u \equiv v$ in $\mathrm{Camb}_{\Omega|_{\langle s \rangle}}$.

In particular, we never have $u \equiv v$ if $u \geq s$ and $v \not\geq s$ or vice versa.

November 27 – Beyond the Tits cone

Let W be an infinite Coxeter group. Then W is not a lattice, since joins may not exist at all. In this lecture, we will explore attempts to define a larger lattice, with W sitting at the bottom.

There is a dumb answer: Just add one extra element $\hat{1}$ which is bigger than every element of W. This is a complete lattice. The element $\hat{1}$ is the join of all subsets of W which didn't have joins before (and of all subsets of the extended lattice that contain $\hat{1}$); it is also the meet of the emptyset and of the singleton $\{\hat{1}\}$.

Why don't we like this answer?

- It has no relation to the representation theory of quiver path algebras, or preprojective algebras. It has no relation to cluster algebras (which are related to the previous two topics.)
- We expect that there should be an order reversing symmetry of our order, corresponding to $u \mapsto uw_0$ in finite type.
- Lam and Pylyavskyy, "Total positivity for loop groups II: Chevalley generators", provide an excellent candidate for the subset of our larger lattice which can be described as $\bigvee w_i$ where w_i is a sequence $w_1 < w_2 < \cdots$ in affine type. Chen and Labbé "Limit directions for Lorentzian Coxeter systems" generalize this to hyperbolic groups and

Lam and Thomas "Infinite reduced words and the Tits boundary of a Coxeter group" generalize to general Coxeter groups.

There is also an important construction that does **not** work. We can **not** look at regions of $V^\vee \setminus \bigcup_{\beta\in\Phi^+} \beta^\perp$. Note that this is equivalent to looking at sets of roots of the form $\{\beta \in \Phi^+ : \langle\theta, \beta\rangle < 0\}$ for various $\theta \in V^\vee$. (Moreover, switching $<$ to $\leq$ doesn't help.) Let's see why this doesn't work.

Consider type $\tilde{A}_3$, whose Coxeter diagram is a 4-cycle. We should have elements of our hypothetical lattice corresponding to the regions s_1s_2D, s_3s_4D, $s_2s_3(-D)$ and $s_4s_1(-D)$. The set of roots corresponding to s_1s_2D is the inversions of s_1s_2, namely, $\{\alpha_1, \alpha_1+\alpha_2\}$. Similarly, s_3s_4D corresponds to $\{\alpha_3, \alpha_3+\alpha_4\}$, and the regions $s_2s_3(-D)$ and $s_4s_1(-D)$ correspond to $\Phi^+ \setminus \{\alpha_2, \alpha_2+\alpha_3\}$ and $\Phi^+ \setminus \{\alpha_4, \alpha_4+\alpha_1\}$. Note that the sets $\{\alpha_1, \alpha_1+\alpha_2\}$ and $\{\alpha_3, \alpha_3+\alpha_4\}$ are contained in both the sets $\Phi^+ \setminus \{\alpha_2, \alpha_2+\alpha_3\}$ and $\Phi^+ \setminus \{\alpha_4, \alpha_4+\alpha_1\}$. So, if we had some hypothetical set X which was the join of s_1s_2 and s_3s_4, we would want to have α_1, $\alpha_1+\alpha_2$, α_3, $\alpha_3+\alpha_4 \in X$ but α_2, $\alpha_2+\alpha_3$, α_4, $\alpha_4+\alpha_1 \notin X$. But it is impossible to have

$$\langle\theta, \alpha_1+\alpha_2\rangle,\ \langle\theta, \alpha_3+\alpha_4\rangle < 0 \text{ and } \langle\theta, \alpha_2+\alpha_3\rangle,\ \langle\theta, \alpha_1+\alpha_4\rangle \geq 0$$

since $(\alpha_1+\alpha_2)+(\alpha_3+\alpha_4) = (\alpha_2+\alpha_3)+(\alpha_1+\alpha_4)$.

We can carry out a similar analysis with $(s_1s_2s_3s_4)^N(D)$, $(s_3s_4s_1s_2)^N(D)$, $(s_2s_3s_4s_1)^N(-D)$ and $(s_4s_1s_2s_3)^N(-D)$ and conclude that there should be some X in our hypothetical lattice which is greater than all $(s_1s_2s_3s_4)^N(D)$ and $(s_3s_4s_1s_2)^N(D)$, and less than all $(s_2s_3s_4s_1)^N(-D)$ and $(s_4s_1s_2s_3)^N(-D)$. If X is then to be a set of roots, we must have

$$\bigcup_N \left(\mathrm{inv}((s_1s_2s_3s_4)^N) \cup \mathrm{inv}((s_3s_4s_1s_2)^N)\right) \subseteq X \subseteq \Phi^+ \setminus \bigcup_N \left(\mathrm{inv}((s_2s_3s_4s_1)^N) \cup \mathrm{inv}((s_4s_1s_2s_3)^N)\right).$$

It turns out that the two sides of the inequality are equal! So there is a set of roots which really wants to be in our lattice.

Proceeding in this manner, one can build lots of sets which want to be in our lattice. I will describe two attempts to describe them, due to Dyer and Viard.

Let Φ be a root system and let L be a two-dimensional subspace for which $|L \cap \Phi^+| \geq 2$. The roots $L \cap \Phi^+$ live in a half space of the two dimensional space L, so they are ordered by angle, up to reversal. While we won't need it, it is extremely valuable for context to know:

Lemma (Dyer)**.** With L as above, either $L \cap \Phi^+$ is finite, or else $L \cap \Phi^+$, in which case it has order type $\frac{0}{1} < \frac{1}{2} < \frac{2}{3} < \cdots < \frac{3}{2} < \frac{2}{1} < \frac{1}{0}$.

The following definition is due to Dyer[15]: Let $B \subseteq \Phi^+$. We say that B is ***biclosed*** if, for all L as above, $L \cap B$ is either an initial or a final segment of $L \cap \Phi^+$.

Dyer conjectures that biclosed sets, with respect to containment, form a complete lattice. The evidence is the following:

- If W is finite, then biclosed sets are inversion sets. More generally, finite biclosed sets are inversion sets. This is a good exercise.
- Dyer asserts he has checked this in affine type. David Speyer and Grant Barkley have checked $\tilde{A}$ and $\tilde{C}$, and maybe also the other classical affine types.

[15] "On the weak order of Coxeter groups". See also Hohlweg and Labbè, "On inversion sets and the weak order in Coxeter groups".

- Thomas McConville and collaborators have a number of papers that look at biclosed sets in certain finite subsets of infinite root systems. See McConville "Lattice structure of grid-Tamari orders", Garver and McConville "Oriented flip graphs of polygonal subdivisions and noncrossing tree partitions" and Garver, McConville and Mousavand, "A categorification of biclosed sets of strings". Everything in these papers looks like a quotient of an infinite lattice, where we have discarded all but finitely many shards as in our Third Example of a quotient of S_n.

I'd also like to publicize a lesser known work: Francois Viard was a student of Biagioli and Chapoton who seems to have left math. His thesis, "Des graphes orients aux treillis complets : une nouvelle approche de l'ordre faible sur les groupes de Coxeter" introduces a fascinating new approach to studying weak order. (Don't be scared, only the title and introduction are in French; the rest is in English.)

He makes the following definition: Let G be a directed acyclic graph where every vertex has (finite) even out degree, and where there are no infinite chains of the form $\bullet \to \bullet \to \bullet \to \cdots$. Let V be the set of vertices of G and, for $v \in V$, let $\mathrm{Out}(v) = \{w \in V : v \to w\}$.

Define a subset C of V to be ***consensus*** if

- If $v \in V$ then $|C \cap \mathrm{Out}(v)| \geq \frac{1}{2}|\mathrm{Out}(v)|$.
- If $v \notin V$ then $|C \cap \mathrm{Out}(v)| \leq \frac{1}{2}|\mathrm{Out}(v)|$.

Or, stated in the contrapositive, if $|C \cap \mathrm{Out}(v)| < \frac{1}{2}|\mathrm{Out}(v)|$ then $v \notin V$; if $|C \cap \mathrm{Out}(v)| > \frac{1}{2}|\mathrm{Out}(v)|$ then $v \in V$ and, if $|C \cap \mathrm{Out}(v)| = \frac{1}{2}|\mathrm{Out}(v)|$ then we have a free choice.

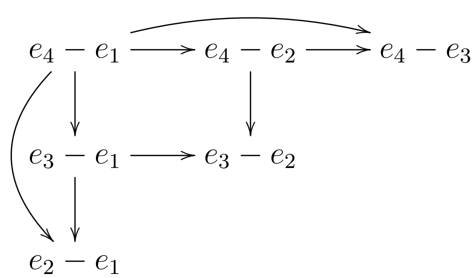
Viard calls these sets ***balanced***, but I think that the metaphor of V as voters and C as a consensus is too good to pass up.

As an example, let G be $1 \leftarrow 2 \to 3$. Then the consensus sets are $\emptyset$, $\{1\}$, $\{3\}$, $\{1, 2\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. Hopefully, the reader by now recognizes these as the inversion sets in A_2. The task of suggesting three public figures whose political views have this behavior is left to the reader.

As we checked on a problem set for finite V, the set of consensus sets is always a lattice.

Viard suggests that we may be able to put a directed graph structure on Φ^+ for which the consensus sets are the biclosed sets.

Indeed, in types A, D and E, we can put $\gamma \to \beta$ if β cuts γ. As an example, here is the graph for A_3:



Let's prove that this works. In other words, given a subset X of Φ^+ in types A, D or E, let's show that it X is biclosed if and only if it is consensus.

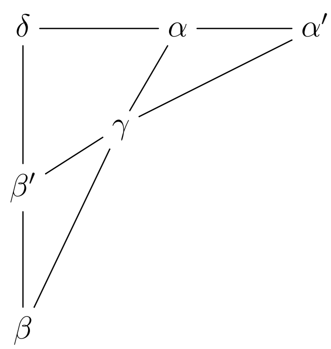
Let $\beta \in \Phi^+$. Then the roots which cut β can be organized into pairs $\{\alpha, \gamma\}$ with $\alpha + \gamma = \beta$ in each pair (since all the non-commutative rank two parabolics are A_2's in types ADE).

Suppose that X is biclosed. Consider any $\beta \in X$, we will show that the majority of $\mathrm{Out}(\beta)$ is in X. Indeed, pair off the elements as above and, in each pair (α, γ), the biclosed condition shows that at least one of α and γ must be in X. The case where $\beta \notin X$ is similar.

Suppose that X is consensus. Choose a total order on Φ^+ refining the cutting partial order (as in our proof of congruence uniformity). We will show by induction on γ that $X \cap \{\alpha, \gamma, \beta\}$ obeys the biclosed condition for each A_2-root-subsystem with β in the middle.

Suppose to the contrary that this fails for some $\{\alpha, \gamma, \beta\}$. We'll do the case that $X \cap \{\alpha, \gamma, \beta\} = \{\gamma\}$; the case where the intersection is $\{\alpha, \beta\}$ is similar.

The consensus condition implies there must be some other $\{\alpha', \gamma, \beta'\}$ with $\alpha' + \beta' = \gamma$ and $\{\alpha', \gamma, \beta'\} \subseteq X$. Now, look at the rank 3-subsystem spanned by $\{\alpha, \alpha', \beta, \beta', \gamma\}$. This must be an A_3, because all the rank two parabolics in type ADE are A_3, $A_2 \times A_1$ or $A_1 \times A_1 \times A_1$, and the latter two cases don't have five roots obeying $\alpha + \beta = \alpha' + \beta' = \gamma$. Then, up to symmetries of the situation, we can assume the sixth positive root in this A_3 is $\alpha - \alpha' = \beta' - \beta =: \delta$. So the roots are laid out as in the diagram below:



By induction, we know the biclosure condition holds for the root subsystems $(\delta, \alpha, \alpha')$ and (δ, β', β). But then the fact that $\alpha \notin X$ and $\alpha' \in X$ implies that $\delta \notin X$, and the fact that $\beta' \in X$ and $\beta \notin X$ implies that $\delta \in X$. This is a contradiction.

December 2 – Introduction to Bruhat order

Today, we move on from weak order and consider Bruhat order, a different partial order on Coxeter groups. First, a few notes on notation:

- From here on, we'll be using $\leq$ by default to refer to Bruhat order. If we want to refer to weak order, we'll use $\leq_L$ or $\leq_R$, according to whether it's left or right weak order.
- We've previously used $s_1, \dots, s_\ell$ as a list of the simple reflections of a Coxeter group in a fixed order; but in what follows, to avoid descending into an abyss of nested indexing, we'll use these to refer to arbitrary simple reflections.
- When we refer to a "subword" of a word, we allow the letters to be nonconsecutive, but they must appear in order. "CAN" is a subword of "MICHIGAN"; "GAIN" is not.

Definition. Let W be a Coxeter group, $u, w \in W$. Then we say $u \leq w$ ***in Bruhat order*** if any of the following equivalent conditions hold:

(1) There is a reduced word $s_1 s_2 \cdots s_\ell$ for w with a subword $s_{k_1} \cdots s_{k_m} = u$.
(2) For *any reduced word* $s_1 s_2 \cdots s_\ell$ for w, there exists a subword $s_{k_1} \cdots s_{k_m} = u$.
(3) For *any word* $s_1 s_2 \cdots s_\ell$ for w, there exists a subword $s_{k_1} \cdots s_{k_m} = u$.
(4), (5), (6) The same as the previous three conditions, respectively, but where the subword is required to be a *reduced* word for u.

Some simple examples are shown in Figures 31 and 32. We can immediately note one difference between Bruhat and weak order: Bruhat order is not a lattice. For example, s_1 and s_2 have no join in either example.

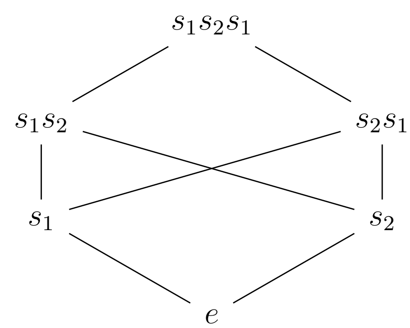


FIGURE 31. Bruhat order in type A_2.

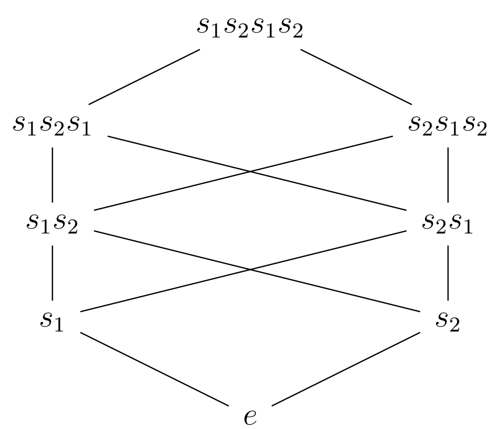


FIGURE 32. Bruhat order in type $I_2(4) = B_2$.

We now set about proving the equivalence of these definitions. First, a lemma:

Lemma. Let $s_1 \cdots s_\ell$ be any word. If there is a subword with product v, then there is a *reduced* subword with product v.

Proof. We can consider without loss of generality the case that $v = s_1 \cdots s_\ell$, ignoring the simple reflections not appearing in the initial word for v. Let $t_1, \ldots, t_\ell$ be the reflection sequence. If the word is not reduced, we must have $t_i = t_j$ for some $i \neq j$. We then claim that removing s_i and s_j from the initial word produces another, shorter word for v. Indeed, we have

$$\begin{aligned} v = t_i t_j v &= t_i(s_1 s_2 \cdots s_j \cdots s_2 s_1)(s_1 \cdots s_\ell) \\ &= t_i(s_1 \cdots \widehat{s}_j \cdots s_\ell) \\ &= (s_1 s_2 \cdots s_i \cdots s_2 s_1)(s_1 \cdots \widehat{s}_j \cdots s_\ell) \\ &= s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_\ell \end{aligned}$$

We can continue doing this until we end up at a subword which is a reduced word for v. □

With this lemma, we then know that (1) ⇔ (4), (2) ⇔ (5), (3) ⇔ (6), since existence of a subword giving u implies existence of a reduced subword giving u. Addtionally, it's straightforward to check that (3) ⇒ (2) ⇒ (1), and we know (2) ⇒ (3) because any word for u will have a reduced subword.

It remains to show that:

Lemma. (1) ⇒ (2): if any reduced word for w contains a subword which is a word for u, they all do.

Proof. Let $w = s_1 \cdots s_\ell = s'_1 \cdots s'_\ell$ be two reduced words for w. We want to show that the set of elements we get as products of subwords is the same for either word.

We know from a previous homework exercise that we can transform the first word into the second just by applying braid moves. Thus it will suffice to show that the set of elements obtained as products of subwords is unchanged by the application of a single braid move

$$\cdots(s_i s_j s_i s_j \cdots)\cdots$$
$$\Downarrow$$
$$\cdots(s_j s_i s_j s_i \cdots)\cdots$$

If we obtain some element as the product of a subword of the top word, we can break that subword down into the reflections coming from the left ellipsis, the alternating s_i's and s_j's, and the right ellipsis. We can carry the reflections from the left and right ellipses down to the other word, so all we need to show is that the elements obtained from subwords of $\underbrace{s_i s_j \cdots}_{\text{length } m_{ij}}$ are the same as those obtained from subwords of $\underbrace{s_j s_i \cdots}_{\text{length } m_{ij}}$. But in either case, we just get all elements of the rank 2 parabolic subgroup generated by s_i and s_j. □

So we have a few different ways of looking at Bruhat order. Definition (1) is particularly useful for showing that $u \leq w$, whereas definitions (2) and (3) are useful for showing the negation.

Now we start building up some tools for working with Bruhat order. This will take a little while, but as an example of what we'll eventually show, here are some characterizations of what covers in this order mean.

Theorem. Let $u, v \in W$. The following are equivalent.

- $u \lessdot v$.
- $\ell(u) = \ell(v) - 1$, and there is a reduced word $s_1 \cdots s_\ell$ for v such that $u = s_1 \cdots \widehat{s_k} \cdots s_\ell$.
- $\ell(u) = \ell(v) - 1$, and for *any* reduced word for v, we can obtain u by deleting a reflection as above.
- $\ell(u) = \ell(v) - 1$, and $u^{-1}v$, $v^{-1}u$, uv^{-1}, or vu^{-1} is a reflection.
- $\ell(u) = \ell(v) - 1$, and $u = tv$, $t \in \mathrm{inv}(v)$.

We draw another contrast with weak order by noting the presence of the condition "$\ell(u) = \ell(v) - 1$" in all of these statements. If we ignore this condition, all of the latter 4 bullet points above hold for the elements e and $s_1 s_2 s_1$ in A_2, but they certainly don't form a cover. The problem is that removing a letter from the *middle* of a word can shrink its length dramatically, in contrast to only working at the ends of a word as we do in weak order.

We start working towards this result with a couple of lemmas.

First, let $w \in W$ and $t \in T$, the set of reflections. Then wD and twD are on opposite sides of $\mathrm{Fix}(t)$, which means t is an inversion of w or tw, but not both. We show:

Lemma. If $t \in \mathrm{inv}(tw)$, then $w < tw$.

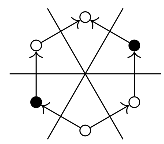
Proof. Choose a reduced word $s_1 s_2 \cdots s_\ell$ for tw. Let t_j be the reflection sequence associated to this word. Because $t \in \mathrm{inv}(tw)$, $t = t_k = s_1 s_2 \cdots s_k \cdots s_2 s_1$ for some k. Then

$$\begin{aligned} w = t(tw) &= ts_1 \cdots s_k \cdots s_\ell \\ &= (s_1 s_2 \cdots s_k \cdots s_2 s_1) s_1 \cdots s_k \cdots s_\ell \\ &= s_1 \cdots \widehat{s}_k \cdots s_\ell \end{aligned}$$

implying $w < tw$. □

Corollary. For all $w \in W$ and $t \in T$, either $tw < w$ or $w < tw$, and we can tell which is true by checking if $t \in \mathrm{inv}(w)$ or $t \in \mathrm{inv}(tw)$, respectively.

We note again that this is in contrast with weak order. The two elements marked in black here are related by a reflection, but are not comparable in right weak order:



Here's another lemma, approaching the last bullet point of the above theorem:

Lemma. If $u \le w$, there exists v with $u < v \le w$, $v = tu$ for some $t \in T$, and $\ell(v) = \ell(u) + 1$.

Proof. Let $w = s_1 \cdots s_q$ be a reduced word. Then there is some reduced word for u obtained by omitting the reflections $s_{k_1}, \ldots, s_{k_r}$, listed in the order they appear in our word for w. From among all such reduced words for u, we choose the one in which s_{k_1} occurs farthest to the right in the word for w.

Now consider the element v given by the word obtained by omitting $s_{k_2}, \ldots, s_{k_r}$ but not s_{k_1} from our word for w. Certainly $v \le w$. We also have $v = tu$, where $t = t_{k_1}$ is the k_1th entry in the reflection sequence of our word for w; this is because t_{k_1} is also the k_1th entry in the reflection sequence of our word for v, as v and w's words do not differ up to that point. By our lemma above, $v = tu$ implies either $u < v$ or $v < u$.

We now rule out the possibility that $v < u$. This will show that $u < v$. Additionally, we know $\ell(v) \le \ell(u) + 1$ because of the way we constructed v from a reduced word for u. So once we know $u < v$, this will imply $\ell(u) < \ell(v)$ and force the equality $\ell(v) = \ell(u) + 1$.

By our lemma above, it will suffice to show that $t \notin \mathrm{inv}(u)$. So suppose for a contradiction that $t \in \mathrm{inv}(u)$. Then t appears in the reflection sequence of our word for u. First, we claim that it must appear after the position we deleted s_{k_1} from. This is because the words (and thus reflection sequences) of w and u are the same up to that point, so if t appeared before the position of s_{k_1}, it would appear a second time in the reflection sequence of w at that position, contradicting that we chose a reduced word for w. Suppose that t appears as the entry at the position of s_p in the reflection sequence for u.

But then, we claim that we can take s_p out of our word for u, put s_{k_1} back in, and the resulting word will also produce u. Because t is the s_p-entry of the reflection sequence, we can write tu with a word obtained by removing s_p from word for u. Then multiplying by t on the left again can be realized by re-inserting s_{k_1}, for the same reason that $tu = v$ (using the fact that p appears to the right of k_1).

We've created a new reduced word for u which agrees with our reduced word for w for more characters. This contradicts the maximality used in our initial choice of a reduced word for u. □

We'll look at some more foundational results in the next few classes. For now, we close by considering: what is the geometric picture here?

As usual, we assume that $D^\circ \neq \emptyset$. Given any element $w \in W$ a point $x \in wD^\circ$, and a reflection $t \in T$, we consider the element $x - tx$. This is a multiple of the coroot $\beta_t^\vee$, by definition. Our above lemma then shows that $tw < w$ precisely when $x - tx$ is a **positive** multiple of $\beta_t^\vee$, that is, when wD° lies on the positive side of $\mathrm{Fix}(t)$ and $t \notin \mathrm{inv}(w)$.

Pictorially, since every reflection induces a relation in Bruhat order, we can view the order overlaid on the hyperplane arrangement as in Figures 33 and 34, where we represent the regions by the points of a W-orbit.

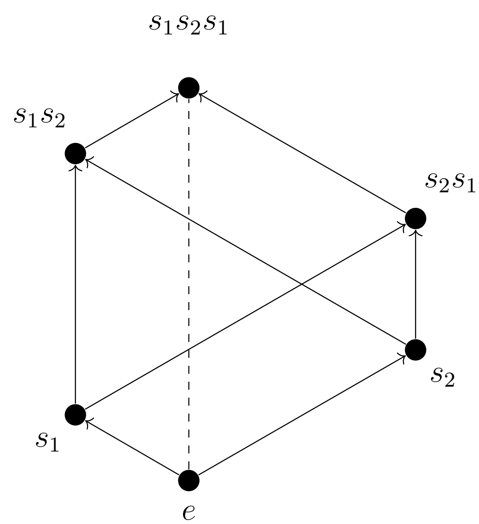


Figure 33. The Bruhat order for A_2, viewed geometrically. Note how the cover relations (together with the non-covering relation $e < s_1s_2s_1$) arise from reflections.

December 4 – Bruhat Order on Parabolic Quotients

Last time, we saw what covers looked like in Bruhat Order: $vt \lessdot v$ if and only if $t \in T$ with $\ell(vt) = \ell(v) - 1$.

What does this look like in type A? All reflections in type A are of the form (ij) with $i < j$. If we have that $w(ij) \lessdot w$, then we have that (ij) is an inversion of w, so $w_i > w_j$ and we need $\ell(w(ij)) = \ell(w) - 1$, which will happen if and only if there is no $k \in (i, j)$ with $w_i > w_k > w_j$. When viewing type A as permutation matrices, we have that this condition will hold if and only if the submatrix with corners (i, w_i) and (j, w_j) has no other nonzero entries besides at these corners.

Now, we turn toward seeing how parabolic subgroups behave with this order. We first recall some basic things we know about parabolic subgroups.

Say the set of generators of W is S. For a subset $J \subseteq S$, we define $W_J = \langle s | s \in J\rangle$, and this is also a Coexter group. We also have the sets ${}^JW = \{w \in W | s$ is a left ascent of w,

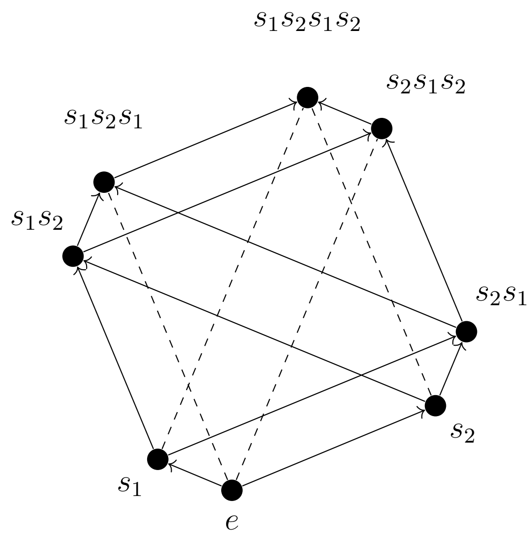


FIGURE 34. The Bruhat order for B_2, viewed geometrically. Again, non-covering relations induced by hyperplanes are depicted by dashed lines.

for all $s \in J\}$, and $W^J = \{w \in Ws$ is a right ascent of w, for all $s \in J\}$. We proved that every $w \in W$ can be factored uniquely as $w = w_J {}^J w = w^J {}_J w$ for some ${}^J w \in {}^J W$, $w^J \in W^J$, ${}_J w, w_j \in W_J$. Moreover, we have that $\ell(w) = \ell(w_J) + \ell({}^J w) = \ell(w^J) + \ell({}_J w)$.

Now, the fact that every w can be written uniquely as $w = w^J {}_J w$ with $w^J \in W^J$ and ${}_J w \in W_J$ means that the set W^J acts as a set of representatives for the cosets in W/W_J. This allows us to give the following definition

Definition. We define the Bruhat order on the quotient W/W_J by saying that $uW_J \leq vW_J$ if the corresponding W^J representatives, u^J, v^J satisfy $u^J \leq v^J$.

Now, this definition is standard, but, obviously, not very good. Why would we need to pass to the quotient and not just work with the set W^J since the order would just be the restriction in this case? This inspired Professor Speyer to give an alternative, equivalent definition that makes more sense in terms of talking about quotients:

Definition. We define the Bruhat order of the quotient W/W_J by saying that $uW_J \leq vW_J$ if there exists $u' \in uW_J$ and $v' \in vW_J$ with $u' \leq v'$.

Now, clearly the first definition implies the second, and after a few lemmas, we will see that the second implies the first.

Lemma (Exchange Lemma)**.** Suppose $u < v$, s is a right ascent of u and a right descent of v. Then $us \leq v$ and $u \leq vs$.

Proof. Choose a reduced word $v = s_1 \dots s_\ell$ with $s_\ell = s$ (we can do this because s is a right descent of v). Now pick a reduced subword $u = s_{i_1} \dots s_{i_m}$, noting that $s_{i_m} \neq s$, since s is a right ascent of u. Then we have that $us = s_{i_1} \dots s_{i_m} s$ is a subword of $s_1 \dots s_\ell$, so $us \leq v$, and $s_{i_1} \dots s_{i_m}$ is a subword of $s_1 \dots s_{\ell-1} = vs$ so $u \leq vs$. $\square$

Lemma. The map $u \to u^J$ is order preserving.

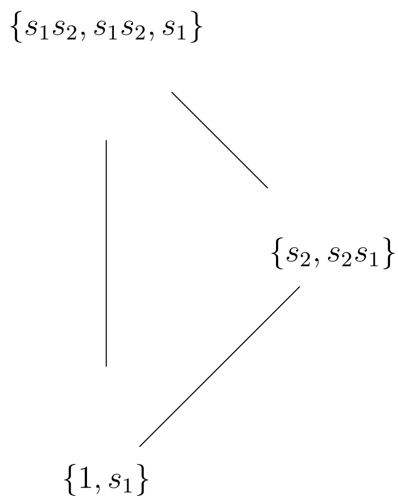


FIGURE 35. Quotient for A_2, with $J = \{s_1\}$

Proof. Let $u \le v$ in W. Want $u^J \le v^J$. We do this by induction on $\ell({}_Jv)$.

Now, note that because $\ell(u^J) + \ell({}_Ju) = \ell(u)$, we have that $u^J \le u \le v$. If $\ell({}_Jv) = 0$, then $v = v^J$ and we are done. Otherwise, v has some right descent $s \in J$. But also, s is a right ascent of u^J, so we must have $u^J \le vs$. By the inductive step, we have that $u^J \le (vs)^J$. But $(vs)^J = v^J$, since they are both in the same coset. □

Now, we can show that the second definition implies the first. Suppose we have uW_J, vW_J, with $u' \in uW_J$, $v' \in vW_J$, and $u \le v$. Then $u^J \le v^J$, so the cosets $uW_J \le vW_J$ in the first definition.

Putting this order in geometric terms, pick a representation of W with the $\alpha_s, \alpha_s^\vee$ linearly independent. Recall that we had the the stabilizer of a vector x with $\langle x, \alpha_s\rangle = 0$ for $s \in J$ and $\langle x, \alpha_s\rangle > 0$ is W_J. This means that we can identify the quotient W/W_J with the orbit Wx. And it turns out that we can view the order relation in terms of the orbit in an equivalent way:

Proposition. Bruhat order on W/W_J is the transitive closure of the relation $ty < y$ if $ty - y \in \mathbb{R}_{>0}\beta^\vee$.

Proof. Suppose we had $vx, t_1vx, \ldots, t_n \ldots t_1vx$ with $t_{k-1} \ldots t_1vx - t_k \ldots t_1vx \in \mathbb{R}_{>0}\beta_{t_k}^\vee$ for all k. Put $w_k = t_k \ldots t_1v$. We know that $w_kx \in w_kD$. We can perturb x to be in D^0, and then the same perturbation after applying w_k will land in w_kD^0. Then the fact that $w_{k-1}x - w_kx \in \beta_{t_k}^\vee$ means that $w_{k-1} > w_k$, so we have $w_1 > w_2 > \cdots > w_n$.

Conversely, suppose $w_1 > 2_2 > \cdots > w_n$, with $w_k = t_kw_{k-1}$. Then we have that $w_kx - w_{k-1}x \in \mathbb{R}_{\ge 0}\beta_T^\vee$. If $w_kx - w_{k-1}x = 0$, then they are equal and we can remove them. Otherwise, we have that $w_{k-1}x > w_kx$, and so we are good. □

In S_n the $W_J's$ are all Young subgroups, $S_k \times S_{n-k}$. In this case we have a bijection between $S_n/S_k \times S_{n-k}$ and $\binom{[n]}{k}$. In this case, we have that $Wx = \{e_{i_1} + \cdots + e_{i_k} | 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ and $(i_1, \ldots, i_k) < (j_1, \ldots, j_k)$ if and only if $i_1 < j_1, , i_2 < j_2, \ldots,$ and $i_k < j_k$.

A special case of this is when we chose $J = [n+1] \backslash \{j\}$. In this case, the quotient is a distributive lattice. This is part of a general theory of "minuscule" parabolic quotients:

Theorem. Let W be a Coxeter group and W_J a parabolic subgroup. The following are equivalent:

(1) W^J is a lattice with respect to Bruhat order.
(2) W^J is a distributive lattice with respect to Bruhat order.
(3) W^J is a distributive lattice with respect to weak order.
(4) $\leq$ and $\leq_R$ coincide on W^J.

See Stembridge, "On the Fully Commutative Elements of Coxeter Groups", Theorem 7.1 (1996) and Proctor "Bruhat lattices, plane partition generating functions, and minuscule representations" (1984). The parabolic quotients described in the question are called the ***minuscule quotients***. If W_1/W_{J_1} and W_2/W_{J_2} are minuscule, then so is $W_1/W_{J_1} \times W_2/W_{J_2}$, so we can reduce to the connected case. Here is the list of connected minuscule quotients. In each case, $[n] \setminus J$ is singleton; we mark it with a $\circ$.

Dynkin diagram	Description as W/W_J	Alternate description
$\bullet$ —— $\bullet$ —— $\cdots$ —— $\circ$ —— $\cdots$ —— $\bullet$	$(A_n/A_{k-1} \times A_n)$	$\binom{[n]}{k}$
$\circ \overset{4}{——} \bullet$ —— $\cdots$ —— $\bullet$	B_n/A_{n-1}	$\{-1, 1\}^n$
$\bullet \overset{4}{——} \bullet$ —— $\cdots$ —— $\circ$	B_n/B_{n-1}	$\{\pm 1, \pm 2, \ldots, \pm n\}$
$\bullet$ —— $\bullet$ —— $\cdots$ —— $\circ$ | $\bullet$	D_n/D_{n-1}	$\{\pm 1, \pm 2, \ldots, \pm n\}$ but 1 and -1 are incomparable
$\bullet$ —— $\bullet$ —— $\cdots$ —— $\bullet$ | $\circ$	D_n/D_{n-1}	$(s_1, \ldots, s_n) \in \{-1, 1\}^n$ where $\prod s_j$ is fixed at 1
$\circ \overset{m}{——} \bullet$, $m \leq \infty$	$I_2(m)/\langle s_1 \rangle$	$\{1, 2, \ldots, m\}$
$\bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet$ —— $\circ$ | $\bullet$	E_6/D_5	
$\bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet$ —— $\circ$ | $\bullet$	E_7/E_6	