BASIC STRUCTURE OF COXETER GROUPS

These are class notes from the first month of Math 665, taught at the University of Michigan, Fall 2017. Class taught by David E Speyer; class notes edited by David and written by the students: Elizabeth Collins-Wildman, Deshin Finlay, Haoyang Guo, Alana Huszar, Gracie Ingermanson, Zhi Jiang and Robert Walker. Thanks also to the Fall 2019 class, Anna Brosowsky, Shelby Cox, Will Dana, Will Newman, Ruilin Shi and Danny Stoll for their comments and suggestions. Further comments and corrections are welcome!

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1. Introduction

This course will study the combinatorics and geometry of Coxeter groups. We start by discussing the finite reflection groups. We are doing this for three reasons:

(1) The classification of finite reflection groups is the same as that of finite Coxeter groups, and the structure of finite reflection groups motivates the definition of a Coxeter group.
(2) One of my favorite things in math is when someone describes a natural sort of object to study and turns out to be able to give a complete classification. Finite reflection groups is one of those success stories.
(3) This will allow us to preview many of the most important examples of Coxeter groups. I'll also take the opportunity to tell you their standard names.

Today we will, without proof, describe all the finite reflection groups.

Let V be a real vector space with a positive definite symmetric bilinear form $\cdot$ (dot product). Let α be a nonzero vector in V. The ***orthogonal reflection*** across $\alpha^\perp$ is the linear map

$$x \mapsto x - 2\frac{\alpha \cdot x}{\alpha \cdot \alpha}\alpha.$$

This fixes the hyperplane $\alpha^\perp$ and negates the normal line $\mathbb{R}\alpha$. An ***orthogonal reflection group*** is a subgroup of $\mathrm{GL}(V)$ generated by orthogonal reflections. We will be classifying the finite orthogonal reflection groups.

Let $W_1 \subset \mathrm{GL}(V_1)$ and $W_2 \subset \mathrm{GL}(V_2)$ be reflection groups. Then we can embed $W_1 \times W_2$ into $\mathrm{GL}(V_1 \oplus V_2)$, sending (w_1, w_2) to $\left[\begin{smallmatrix} w_1 & 0 \\ 0 & w_2 \end{smallmatrix}\right]$. We call a reflection group ***irreducible*** if we can not write it as a product in this way. We will now begin describing all of the irreducible orthogonal reflection groups.

The trivial group: Take $V = \mathbb{R}$ and $W = \{1\}$.

The group of order 2: Take $V = \mathbb{R}$ and $W = \{\pm 1\}$. This group can be called A_1 or B_1.

The dihedral group: Let m be a positive integer and consider the group of symmetries of regular m-gon in $\mathbb{R}^2$. This is generated by reflections across two hyperplanes with angle π/m between them. Calling these reflections σ and τ, we have $\sigma^2 = \tau^2 = (\sigma\tau)^m = 1$. This dihedral group has order $2m$ and can be called $I_2(m)$. We note a general rule of naming conventions – the subscript is always the dimension of V.

The symmetric group: Consider the group S_n acting on $\mathbb{R}^n$ by permutation matrices. The transposition (ij) is the reflection across $(e_i - e_j)^\perp$. This breaks down as $\mathbb{R}(1, 1, \ldots, 1) \oplus (1, 1, \ldots, 1)^\perp$ So the irreducible reflection group is S_n acting on $(1, 1, \ldots, 1)^\perp$. This is called A_{n-1}.

The group $S_n \ltimes \{\pm 1\}^n$: Consider the subgroup of $GL_n(\mathbb{R})$ consisting of matrices which are like permutation matrices, but with a ± 1 in the nonzero positions. This is a reflection group, generated by the reflections over $(e_i - e_j)^\perp$, $(e_i + e_j)^\perp$ and $e_i^\perp$. We give example matrices for $n = 3$ below:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reflection across $(e_1 - e_2)^\perp$ $\quad (e_1 + e_2)^\perp$ $\quad e_1^\perp$

For reasons we will get to eventually, this is both called B_n and C_n.

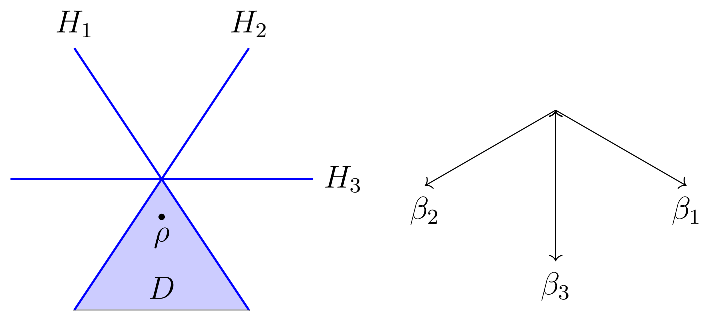


FIGURE 1. Our running example of a hyperplane arrangement

The group $S_n \ltimes \{\pm 1\}^{n-1}$: Consider the subgroup of the previous group where the product of the nonzero entries in the matrix is 1. This is an index two subgroup of B_n, generated by the reflections across $(e_i \pm e_j)^\perp$. It is called D_n.

Collisions of names: We have $A_1 \cong B_1$, $D_1 \cong \{1\}$, $A_1 \times D_1 \cong I_2(1)$, $A_1 \times A_1 \cong D_2 \cong I_2(2)$, $A_2 \cong I_2(3)$, $B_2 \cong I_2(4)$ and $A_3 \cong D_3$. (The last is not obvious.) Some authors will claim that some of these notations are not defined, but if you define them in the obvious ways, this is what you have. Also, $I_2(6)$ has another name, G_2.

We have now listed all but finitely many of the finite orthogonal reflection groups. The remaining cases are probably best understood after we start proving the classification, but we'll try to say something about them. Their names are E_6, E_7, E_8, F_4, H_3 and H_4.

Sporadic regular solids: H_3 is the symmetry group of the dodecahedron and its dual the icosahedron. F_4 is the symmetry group of a regular 4-dimensional polytope called the 24-cell, H_4 is the symmetry group of two regular 4-dimensional polytopes: the 120-cell and the 600-cell. We'll discuss this further in Section 14.

Symmetries of lattices: F_4 is the group of symmetries of the lattice $\mathbb{Z}^4 \cup \left[\mathbb{Z}^4 + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right]$ in $\mathbb{R}^4$. E_8 is the group of symmetries of the eight dimensional lattice

$$\left\{(a_1, a_2, \ldots, a_8) \in \mathbb{Z}^8 \cup \left[\mathbb{Z}^8 + \left(\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}\right)\right] : \sum a_i \in 2\mathbb{Z}\right\}.$$

E_7 is the subgroup of this stabilizing $(1,1,1,1,1,1,1,1)$; E_6 is the subgroup of E_7 stabilizing $(0,0,0,0,0,0,1,1)$. A reflection group which stabilizes a full rank lattice (it need not be the full stabilizer of that lattice) is called ***crystallographic***. The crystallographic finite reflection groups are types A_n, $B_n = C_n$, D_n, E_6, E_7, E_8, F_4 and G_2.

2. Arrangements of hyperplanes

Let V be a finite dimensional $\mathbb{R}$-vector space and let $H_1, H_2, \cdots, H_N$ be finitely many hyperplanes in V. Let $H_i = \beta_i^\perp$ for some vectors $\beta_i \in V^\vee$. The H_i divide V up into finitely many polyhedral cones.

Choose some ρ not in any H_i. Let D° be the open polyhedral cone it lies in and let D be the closure of D°. We'll choose our normal vectors β_i such that $\langle \beta_i, \rho \rangle > 0$. Observe: There can be no linear relation between the β's of the form $\sum c_i \beta_i = 0$ with $c_i \geq 0$ (except $c_1 = c_2 = \cdots = c_N = 0$) since then $\sum c_i \langle \beta_i, \rho \rangle = 0$.

We will call β_i **simple** if β_i is **not** of the form $\sum_{j \neq i} c_j \beta_j$ with $c_j \geq 0$. So

$$D = \{\sigma : \langle \beta_i, \sigma \rangle \geq 0, 1 \leq i \leq n\} = \{\sigma : \langle \beta_i, \sigma \rangle \geq 0, \beta_i \text{ simple }\}.$$

In Figure 1, β_1 and β_2 are simple and β_3 is not.

Now let V have an inner product and let H_i be the reflecting hyperplanes of some finite orthogonal reflection group. Let ρ, D^0, D, H_i and β_i be as before. Call the β_i "positive roots". Let $\alpha_1, \cdots, \alpha_k$ be the simple roots.

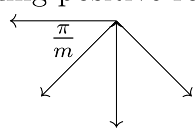
Example. Consider A_{n-1} (the symmetric group S_n) acting on $\mathbb{R}^n$. The reflections are the transpositions (ij); the hyperplanes are $x_i = x_j$ with normal vectors $e_i - e_j$. If we take $\rho = (1, 2, \cdots, n)$, then the positive roots are $e_i - e_j$ for $(i > j)$.

The simple roots are $e_{i+1} - e_i =: \alpha_i$. All the other roots are positive combinations of these, since $e_j - e_i = (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \cdots + (e_{i+1} - e_i)$. So $D = \{(x_1, x_2, \cdots, x_n) : x_1 \leq x_2 \leq \cdots \leq x_n\}$.

Let's review basic linear algebra by computing the angles between these simple roots. Since $\langle \alpha_1, \alpha_3 \rangle = 0$, the angle between α_1 and α_3 is $\frac{1}{2}\pi$. What's the angle between α_1 and α_2? We have $\alpha_1 \cdot \alpha_1 = \alpha_2 \cdot \alpha_2 = (1)^2 + (-1)^2 = 2$, so $|\alpha_1| = |\alpha_2| = \sqrt{2}$ and $\alpha_1 \cdot \alpha_2 = (e_2 - e_1) \cdot (e_3 - e_2) = -1$, so the angle is $\cos^{-1} \frac{-1}{\sqrt{2}\sqrt{2}} = \frac{2}{3}\pi$. These are examples of the lemma we will prove next.

Lemma. In any finite reflection group with $\alpha_1, \cdots, \alpha_k$ as before, the angle between α_i and α_j is of the form $\pi(1 - \frac{1}{m_{ij}})$, $m_{ij} \in \{2, 3, 4, \cdots\}$. Letting s_i be the reflection over $\alpha_i^\perp$, the integer m_{ij} is the order of $s_i s_j$.

Proof. Let θ be the angle between α_i and α_j. Then $s_i s_j$ is rotation by 2θ around $\alpha_i^\perp \cap \alpha_j^\perp$. Letting m be order of $s_i s_j$, we see θ must be of the form $\frac{l}{m}\pi$, where $\mathrm{GCD}(l, m) = 1$. So s_i, s_j generate a copy of the dihedral group of order $2m$. The m reflections in that subgroup are in mirrors $\frac{\pi}{m}$ apart. So the corresponding positive roots look like this:



The simple roots α_i and α_j must be the two at the ends, as the other roots aren't simple. ☐

Corollary. For $i \neq j$, $\langle \alpha_i, \alpha_j \rangle \leq 0$.

Proof. We have $\cos \pi(1 - \frac{1}{m}) \leq 0$ for $m \geq 2$. ☐

Lemma. The simple roots $\alpha_1, \alpha_2, \cdots, \alpha_k$ are linearly independent.

Proof. Suppose $\sum c_i \alpha_i = 0$. We already know we can't have all $c_i \geq 0$. We also can't have all $c_i \leq 0$.

Let $I = \{i : c_i > 0\}$, and define $J = \{j : c_j < 0\}$ analogously. Rewrite the proposed relation as $\sum_{i \in I} c_i \alpha_i = \sum_{j \in J} (-c_j) \alpha_j$, and call this sum γ. But then $\gamma \cdot \gamma = \sum_{i \in I, j \in J} c_i (-c_j) \langle \alpha_i, \alpha_j \rangle \leq 0$. So $\gamma = 0$. But we already showed that there is no nontrivial relationship $\sum c_i \alpha_i = 0$ with all $c_i \geq 0$ or all $c_j \leq 0$. ☐

3. Classification of positive definite Cartan matrices

Let's summarize what we achieved last time. Let V be a finite dimensional vector space with an inner product. Let $W \subseteq GL(V)$ be a finite orthogonal reflection group and $H_1, \ldots, H_n$ be the hyperplanes corresponding to the reflections in W. Choose $\rho \in V - \bigcup_i H_i$. Let β_i be normal to H_i such that $\langle \beta_i, \rho \rangle > 0$ and we let $\alpha_1, \ldots, \alpha_k$ be the simple roots. Let s_i be the reflection in $\alpha_i^\perp$. We showed that the α_i are linearly independent and the angle between α_i and α_j is $\pi\left(1-\frac{1}{m_{ij}}\right)$, where m_{ij} is the order of $s_i s_j$.

Main Idea: Today, we will forget about the reflection group and look at sets of vectors with these properties. It turns out that this is a very limited set.

We begin by normalizing $|\alpha_i|$ to $\sqrt{2}$ so $\langle \alpha_i, \alpha_j\rangle = 2\cos\left[\pi\left(1-\frac{1}{m_{ij}}\right)\right] = -2\cos\frac{\pi}{m_{ij}}$. We call this A_{ij}. We also have $A_{ii} = \langle \alpha_i, \alpha_i\rangle = 2$. The A_{ij} form a symmetric matrix, which we call A.

Example. Let $W = S_n = A_{n-1}$. So $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$. We have

$$A = \begin{bmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{bmatrix}.$$

Proposition. A is positive definite.

Proof. Let $\vec{c} \in \mathbb{R}^k_{\neq 0}$. Then, $\vec{c}^{\,T} A\vec{c} = \sum_{i,j} c_i c_j \langle \alpha_i, \alpha_j\rangle = \left\langle \sum_i c_i\alpha_i, \sum_j c_j\alpha_j\right\rangle \ge 0$. There is equality here if and only if $\sum c_i\alpha_i = 0$, but as the α_i are linearly independent and $\vec{c} \neq 0$, this doesn't happen. Thus, $\vec{c}^{\,T} A\vec{c} > 0$. □

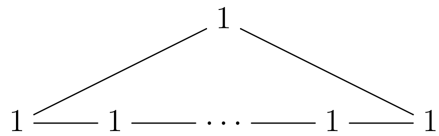
We encode A in a graph Γ: The vertices are 1, …,k and we include an edge (i,j) if $m_{i,j} \ge 3$ (so $A_{ij} < 0$) and we label those edges with m_{ij} if $m_{ij} > 3$. These are called ***Coxeter diagrams***. We'll classify the connected Coxeter diagrams.

Before we begin, we remind the reader of a few key values:

m_{ij}	A_{ij}
2	0
3	-1
4	$-\sqrt{2} \approx -1.41$
5	$-\frac{1+\sqrt{5}}{2} \approx -1.62$
6	$-\sqrt{3} \approx -1.73$

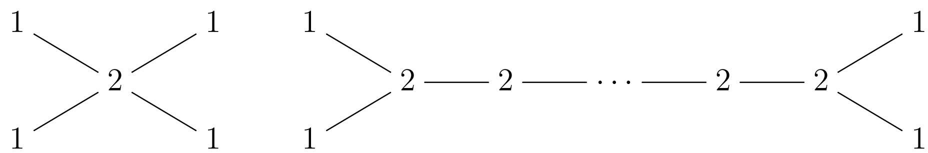
Since positive definite matrices have all submatrices positive definite, Γ cannot contain an induced subgraph corresponding to a non-positive definite matrix. We will thus begin to list non-positive definite graphs. To do this, we use the following convention: If a matrix A isn't positive-definite, there is a nonzero vector c such that $c^T Ac \le 0$. We will label the vertex i with c_i in the below graphs to illustrate why each graph is not positive definite, and write the value of $c^T Ac$ to the right of these graphs.

Observe that, in each of our examples, the vector we have used has positive coordinates. This means that any submatrix whose entries are termwise dominated by those of these graphs is also not positive definite. In particular, these graphs are excluded not only as induced subgraphs, but as subgraphs altogether.



$c^T Ac = 0$

Since Γ cannot contain this subgraph, Γ is a tree.

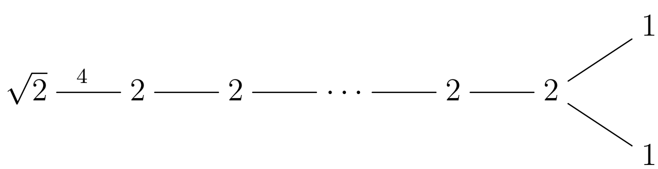


$c^T Ac = 0$

Since Γ cannot contain these subgraphs or any graph whose edge labels dominate them, Γ has no vertices of degree ≥ 4, and at most one of degree 3.

$c^T Ac = 0$

Since Γ cannot contain this subgraph or any graph whose edge labels dominate this, Γ has at most one edge with $m_{ij} \geq 4$.



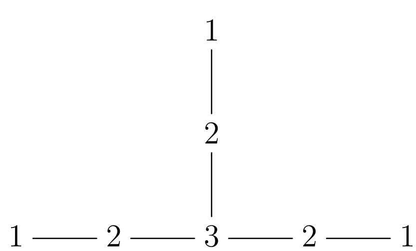
$c^T Ac = 0$

Since Γ cannot contain this subgraph or any graph whose edge labels dominate this, Γ does not have both an edge with $m_{ij} \geq 4$ and a vertex of degree 3.

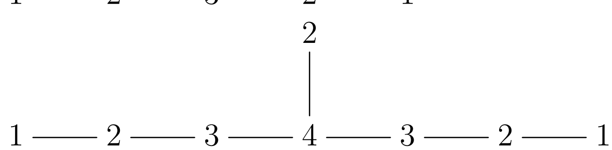
At this point, we know that Γ is either

(1) Three paths with a common endpoint, and all edges with $m_{ij} = 3$ or
(2) A single path, with at most one edge having $m_{ij} > 3$.

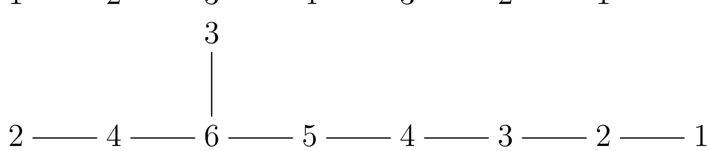
The following excluded graphs rule out all but finitely many three path cases:



$c^T Ac = 0$



$c^T Ac = 0$



$c^T Ac = 0$

The following excluded graphs rule out all but finitely many single path cases.

$$2 \text{ --- } 4 \overset{4}{\text{---}} 3\sqrt{2} \text{ --- } 2\sqrt{2} \text{ --- } \sqrt{2} \qquad c^T A c = 0$$

$$1 \text{ --- } 2 \overset{5}{\text{---}} 2 \text{ --- } 1 \qquad c^T A c = 8 - 4\sqrt{5} < 0$$

$$3 \overset{5}{\text{---}} 4 \text{ --- } 3 \text{ --- } 2 \text{ --- } 1 \qquad c^T A c = 26 - 12\sqrt{5} < 0$$

$$\sqrt{3} \overset{6}{\text{---}} 2 \text{ --- } 1 \qquad c^T A c = 0$$

Remark. Almost all of the graphs above are positive semidefinite, meaning that we can achieve $c^T Ac = 0$ but not $c^T Ac < 0$. Moreover, their kernel is one dimensional. Thus, we are forced to use the precise weights shown here. The exceptions are $1 \text{ --- } 2 \overset{5}{\text{---}} 2 \text{ --- } 1$ and $3 \overset{5}{\text{---}} 4 \text{ --- } 3 \text{ --- } 2 \text{ --- } 1$. In these cases, the matrix has signature $+++-$ and $++++-$, and the weights given were chosen by rounding the eigenvector of negative eigenvalue to the nearest integer vector and checking that the resulting c worked.

Although I say that there is no choice as to what weights I use in the positive semidefinite case, that is a little misleading. I chose to normalize $|\alpha| = \sqrt{2}$ which means, in the vocabulary of the next section, that $\alpha = \alpha^\vee$. If I allow myself the more general setting where α and $\alpha^\vee$ are proportional but not equal, I gain the freedom to choose their ratio, and I could use that freedom to remove the square roots from the figures above.

The surviving graphs are the (connected) Coxeter diagrams! We list them in Table 1.

There are some collisions of names. Some books have fixed conventions about which one of these names to use in each case, but I see no reason not to have two names for the same object.

$$B_n = C_n \quad A_2 = I_2(3) \quad B_2 = I_2(4) \quad G_2 = I_2(6) \quad A_3 = D_3.$$

Reversing the Process: We have now shown that every reflection group generates one of these Coxeter diagrams, but in order to conclude that this is a classification, we need to show that this process can be reversed and is a bijection. It is clear that we can go from the graph Γ to the matrix A. Then, we have the following:

Theorem. For any positive definite symmetric matrix A, there exist vectors $\alpha_1, \alpha_2, ..., \alpha_n$ that are linearly independent such that $\langle \alpha_i, \alpha_j \rangle = A_{ij}$.

Proof. Write $A = U^T DU$ with U orthogonal, D diagonal. Then, $D = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. Let $X = \mathrm{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})U$. Then, $A = X^T X$ and the columns of X have the desired property. □

We can thus go from A to some $\alpha_1, \ldots, \alpha_k$, with corresponding reflections $s_i(x) = x - 2\frac{\langle \alpha_i, x\rangle}{\langle \alpha_i, \alpha_j\rangle}\alpha_i$. These then generate some group W_{output}.

However: The following issues still remain

1. (1) Is W_{output} necessarily finite?
2. (2) If we start with a group W_{input}, run the first algorithm to get a graph Γ and then run it backwards to get W_{output}, does $W_{\text{output}} = W_{\text{input}}$? As a special case, if $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then $W_{\text{output}}(\Gamma)$ is clearly the product of $W_{\text{output}}(\Gamma_1)$ and $W_{\text{output}}(\Gamma_2)$, but it isn't clear that the input group decomposes as a product. So addressing this issue will justify our choice to focus on connected graphs.
3. (3) If we change ρ, do we always get the same Coxeter diagram Γ?
4. (4) If we start with $\alpha_i, \ldots, \alpha_k$, build W_{output} and take a ρ with $\langle \rho, \alpha_i \rangle > 0$, will $\alpha_i, \ldots, \alpha_k$ be the simples we get?

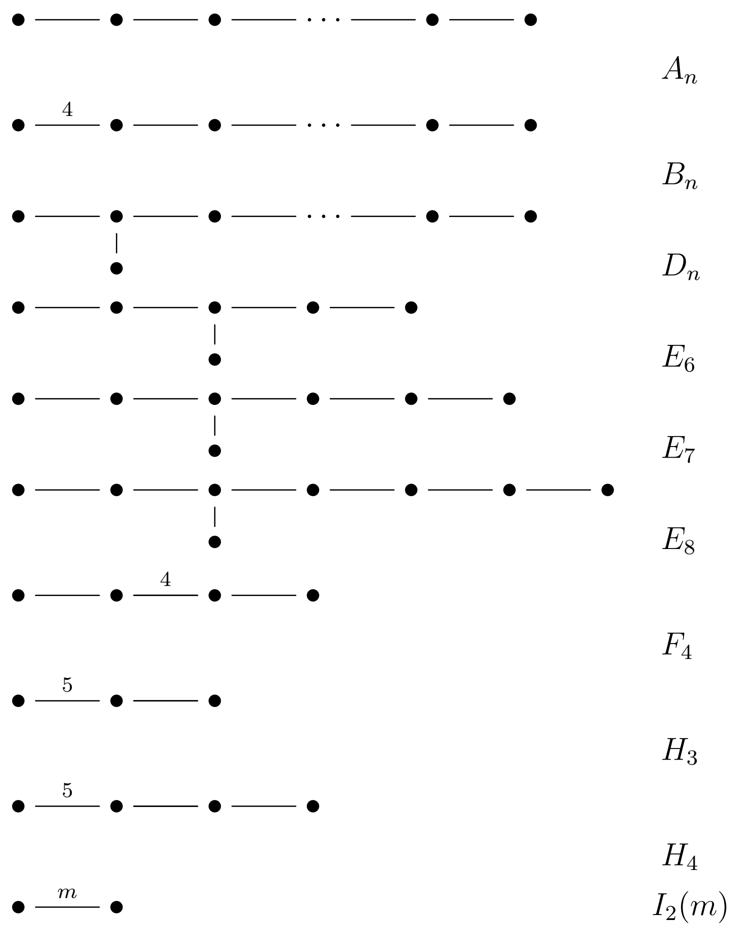


TABLE 1. The finite Coxeter groups

We will show in Section 11 that the answer to all these questions is yes. However, before we answer these questions, we will build up the vocabulary to study infinite Coxeter groups, so that we can prove more general theorems which address these questions in the finite case.

4. Non-orthogonal reflections

Let V be a finite dimensional real vector space and let $V^\vee$ be its dual. We call an element $\sigma \in \mathrm{GL}(V)$ is a reflection if $\sigma^2 = \mathrm{Id}$ and V^σ is of codimension one. Then reflections look like $\sigma(x) = x - \langle \alpha^\vee, x\rangle \alpha$ for some $\alpha \in V$, $\alpha^\vee \in V^\vee$ with $\langle \alpha^\vee, \alpha\rangle = 2$. Note that this formula negates $\mathbb{R}\alpha$ and fixes $(\alpha^\vee)^\perp$.

The vectors α and $\alpha^\vee$ are determined by σ up to rescaling of the form

$$\alpha \mapsto c\alpha, \ \alpha^\vee \mapsto c^{-1}\alpha^\vee.$$

If V is equipped with a nondegenerate symmetric bilinear form, giving an identification $V \cong V^\vee$, then σ is orthogonal if and only if α and $\alpha^\vee$ are proportional. In that case, we will have $\alpha^\vee = \frac{2\alpha}{\langle \alpha,\alpha\rangle}$.

A **reflection group** is defined as a subgroup of $\mathrm{GL}(V)$ generated by reflections.

Lemma. If G is a finite subgroup of $\mathrm{GL}(V)$, then G preserves a positive definite symmetric bilinear form.

Proof. Take any positive-definite bilinear form (,). Set

$$\langle \vec{v}, \vec{w} \rangle := \frac{1}{G} \sum_{g \in G} (g\vec{v}, g\vec{w}).$$

Then ⟨ , ⟩ is positive definite and G-invariant. □

So all finite reflection groups are orthogonal; we will only get interesting new examples in the infinite case.

Remark. Allowing α and $\alpha^\vee$ to be distinct can simplify formulas even in the positive definite case. In the conventions of the last section, the roots of B_n are $\pm e_i \pm e_j$ and $\pm\sqrt{2}e_k$. If we instead take the roots to be $\{\pm e_i \pm e_j, \pm e_k\}$ and the co-roots to be $\{\pm e_i \pm e_j, \pm 2e_k\}$, then we can compute in B_n without any square roots of 2. Indeed, well chosen scalings can remove the square roots in all the crystallographic cases: A, B, C, D, E, F and G.

This explains why we have the double name $B_n = C_n$; we could have chosen the roots to be $\{\pm e_i \pm e_j, \pm 2e_k\}$ and the co-roots to be $\{\pm e_i \pm e_j, \pm e_k\}$ instead. The former choice is B_n and the latter is C_n.

Cartan originally chose these names when classifying simple Lie algebras, and in that context we get specific roots and coroots, with A_{ij} always integral. The Lie algebras B_n and C_n are honestly different, but they have the same associated reflection group. This explains why G_2 gets a special name: It showed up in Cartan's classification, whereas $I_2(m)$ for $m \notin \{2, 3, 4, 6\}$ did not, because for those other values of m we cannot make A_{ij} integral. This is also why H and I are at the end of the list; they were added later.

5. The geometry of two reflections

Let σ and τ be two reflections with

$$\sigma(x) = x - \langle \alpha^\vee, x \rangle \alpha, \qquad \tau(x) = x - \langle \beta^\vee, x \rangle \beta,$$

and $\langle \alpha^\vee, \alpha \rangle = \langle \beta^\vee, \beta \rangle = 2$. Let K be the cone $\{x \in V^\vee : \langle \alpha, x \rangle,\ \langle \beta, x \rangle > 0\}$.

The key fact that we will need is the following:

Theorem. Suppose that $\langle \beta^\vee, \alpha \rangle,\ \langle \alpha^\vee, \beta \rangle \le 0$ and $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle$ is either of the form $4\cos^2 \frac{\pi}{m}$, or else is ≥ 4. If α and β are not proportional, then the images of K under the subgroup generated by σ and τ are disjoint.

If α and β are proportional, then they must point in opposite directions, since $\langle \alpha^\vee, \alpha \rangle$ and $\langle \alpha^\vee, \beta \rangle$ have different signs. So, in this case, K is empty.

The case where $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4$ is a bit annoying, so postpone that one until the end.

The vectors $(\alpha^\vee, \beta^\vee)$ and (α, β) pair by $\begin{bmatrix} 2 & \langle \alpha^\vee, \beta \rangle \\ \langle \beta^\vee, \alpha \rangle & 2 \end{bmatrix}$, which is nonsingular since we assume $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \neq 4$. So V decomposes as $\mathrm{Span}(\alpha, \beta) \oplus \mathrm{Span}(\alpha^\vee, \beta^\vee)^\perp$. The action on the second summand is trivial, so we consider the action on $\mathrm{Span}(\alpha, \beta)$. In the (α, β)-basis, we have

$$\sigma = \begin{bmatrix} -1 & -\langle \alpha^\vee, \beta \rangle \\ 0 & 1 \end{bmatrix} \ \tau = \begin{bmatrix} 1 & 0 \\ -\langle \beta^\vee, \alpha \rangle & -1 \end{bmatrix}$$

and

$$\sigma\tau = \begin{bmatrix} -1 + \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle & \langle \alpha^\vee, \beta \rangle \\ -\langle \beta^\vee, \alpha \rangle & -1 \end{bmatrix}$$

We thus see that $\sigma\tau$ has determinant 1 and trace $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle - 2$.

Case 0: If $\langle \alpha^\vee, \beta \rangle = \langle \beta^\vee, \alpha \rangle = 0$, then σ and τ act on α and β by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ respectively, so σ and τ commutate and generate a copy of the $I_2(2) = A_1 \times A_1$ hyperplane arrangement.

Case I: Suppose that $0 < \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle < 4$; put $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4\cos^2\theta$ for some $\theta \in (0, \pi/2)$. So $\mathrm{Tr}(\sigma\tau) = 4\cos^2\theta - 2 = 2\cos(2\theta)$. We see that the characteristic polynomial of $\sigma\tau$ is $(\lambda - e^{i2\theta})(\lambda - e^{-i2\theta})$. Since $e^{i2\theta} \neq e^{-i2\theta}$, we know that $\sigma\tau$ is conjugate to a rotation by 2θ.

If $\theta = \frac{\ell\pi}{m}$ with $\mathrm{GCD}(\ell, m) = 1$, then σ and τ generate a dihedral group of order $2m$; if θ/π is irrational, then σ and τ generate an infinite group. In particular, if $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4\cos^2\frac{\pi}{m}$, then $\sigma\tau$ is a rotation by $2\pi/m$, the wedge K has central angle π/m and the group $\langle \sigma, \tau \rangle$ permutes $2m$ copies of this wedge.

For future reference note that, if m is even, then the orbits of α and β under $\langle \sigma, \tau \rangle$ each have m elements, and the vectors in the α orbit are not proportional to those in the β orbit. If m is odd, the orbits of each of α and β have $2m$ vectors, and the vectors in the β orbit are proportional to the vectors in the α orbit. The ratio of proportionality is $\sqrt{\frac{\langle \alpha^\vee, \beta \rangle}{\langle \beta^\vee, \alpha \rangle}}$.

Case II: Suppose that $4 < \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle$. Put $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4\cosh^2\phi$. (Recall that the hyperbolic cosine is $\cosh\phi = \frac{e^\phi + e^{-\phi}}{2}$.) So $\mathrm{Tr}(\sigma\tau) = 2\cosh(2\phi)$. So the eigenvalues of $\sigma\tau$ are $e^{\pm 2\phi}$, and thus $\sigma\tau$ has infinite order. More specifically, the eigenspaces of $\sigma\tau$ divide $\mathrm{Span}(\alpha, \beta)$ into 4-cones. The $\langle \sigma, \tau \rangle$ orbit of K covers one of these cones, dividing it into infinitely many non-overlapping wedges. The reader who has studied relativity will recognize $\left[\begin{smallmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{smallmatrix}\right]$ as the matrix of a Poincare transformation; the analogue of a rotation in one space and one time dimension.

Case III: We finally turn back to the case where $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle = 4$.

If α and β are linearly independent, then $\sigma\tau$ acts on the α, β basis by a matrix of the form $\left[\begin{smallmatrix} 3 & 2x \\ -2x^{-1} & -1 \end{smallmatrix}\right]$, hence of Jordan form $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. The translates of K form adjacent cones, filling up a half space. □

We close with some pictures. The actions of σ and τ are depicted by the red and blue arrows, and the reflecting hyperplanes are shown in black. In Figure 2, we have $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle = 2 = 4\cos^2\frac{\pi}{4}$. In Figure 3, we have $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle = 9$.

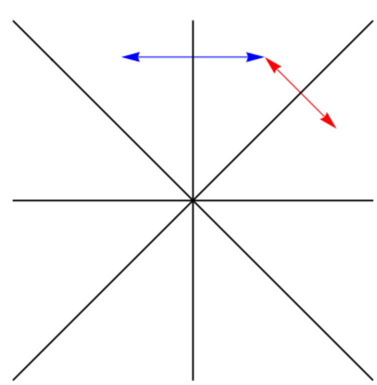


FIGURE 2. The reflecting hyperplanes for the dihedral group $I_2(4)$

We draw several figures in the affine case, when $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha, \rangle = 4$. First, suppose that α and β are not proportional, but $\alpha^\vee$ and $\beta^\vee$ are. In this case, the action in V looks like the left hand side of Figure 4 and the action in $V^\vee$ looks like the right hand side.

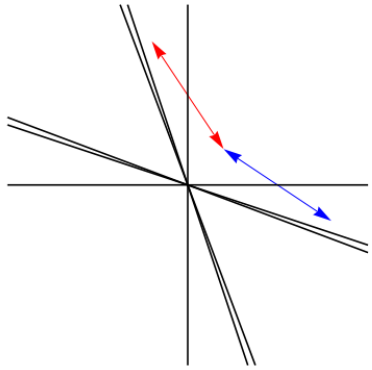


FIGURE 3. The reflecting hyperplanes for a hyperbolic infinite dihedral group

If neither α, β nor $\alpha^\vee$, $\beta^\vee$ are proportional, then $\dim V$ must be at least 3. The action of $\langle \sigma, \tau \rangle$ preserves planes parallel to $\mathrm{Span}(\alpha, \beta)$. Figure 5 depicts the action on such an affine plane.

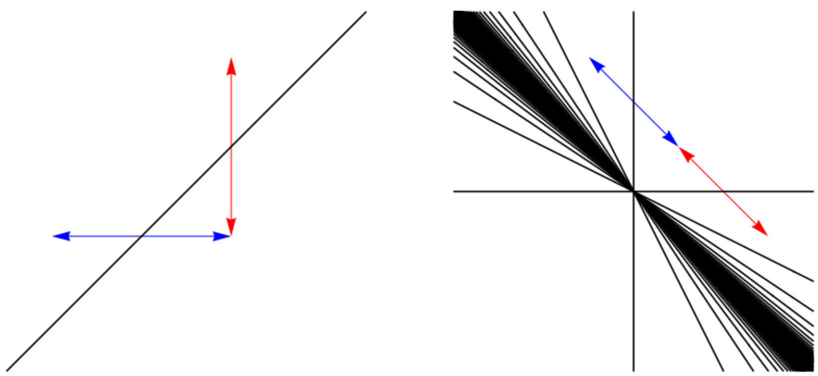


FIGURE 4. An affine infinite dihedral group, acting on V and $V^\vee$

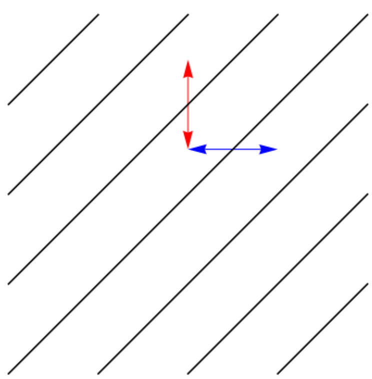


FIGURE 5. An affine slice through a three dimensional representation of the infinite dihedral group

6. Coxeter Groups, Cartan matrices, roots and hyperplanes

The combinatorial data to give a Coxeter group is a collection of integers m_{ij} for $1 \leq i, j \leq n$, with $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$. The Coxeter group W is the group with generators s_i for $1 \leq i \leq n$ and relations $(s_i s_j)^{m_{ij}} = 1$. In particular, each s_i has order 2.

A Cartan matrix A, for our Coxeter group W, is an $n \times n$ matrix satisfying

$$\begin{aligned} A_{ii} &= 2, && \\ A_{ij} &= 0 && \text{if } m_{ij} = 2, \\ A_{ij}A_{ji} &= 4\cos^2 \tfrac{\pi}{m_{ij}},\ A_{ij}, A_{ji} < 0 && \text{if } 3 \leq m_{ij} < \infty, \text{ and} \\ A_{ij}A_{ji} &\geq 4,\ A_{ij}, A_{ji} < 0 && \text{if } m_{ij} = \infty. \end{aligned}$$

If V and $V^\vee$ are dual vector spaces with $\alpha_1, \alpha_2, \ldots, \alpha_k \in V$ and $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_k^\vee \in V^\vee$, we say that the pair by A if $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$. In this case, we get actions of W on both V and $V^\vee$ with $s_i(x) = x - \langle \alpha_i^\vee, x\rangle \alpha_i$, and $s_i(x^\vee) = x^\vee - \langle x^\vee, \alpha_i \rangle \alpha_i^\vee$. The α_i and $\alpha_i^\vee$ are called the ***simple roots*** and ***simple coroots***.

Let $D = \{x^\vee \in V^\vee | \langle x^\vee, \alpha_i \rangle \geq 0\}$, and $D^\circ = \{x^\vee \in V^\vee | \langle x^\vee, \alpha_i \rangle > 0\}$. We will almost always impose that D° is non-empty.

The set of ***roots*** is defined by $\Phi = W \cdot \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset V$.

We write T for the set of elements of W of the form $w s_i w^{-1}$. (Think T for "transposition".) Each t in T acts on V by a reflection; if $t = w s_i w^{-1}$ then t acts by $x \mapsto x - \langle \beta_t^\vee, x \rangle \beta_t$ where $\beta_t = w\alpha_i$ and $\beta_t^\vee = w\alpha_i^\vee$. Later, we will see that every element of W which acts by a reflection on V is in T.

Remark. We have set up our vocabulary in such a general manner that β and $c\beta$ could both be roots, for some $c > 1$. Also, for this reason, the notations β_t and $\beta_t^\vee$ should only be considered up to scalar. In most practical cases, this issue doesn't come up. The precise criterion is the following: Suppose that, whenever m_{ij} is a finite odd integer, we have $A_{ij} = A_{ji}$. Then, if β_1 and β_2 are proportional roots, we must have $\beta_1 = \pm\beta_2$. We'll be ready to prove this in Section 13.

In the mean time, note that if our goal is to study Coxeter groups rather than root systems, we can just choose to take $A_{ij} = A_{ji}$ and be safe. If we further assume that the α_i are a basis of V, there is an easy proof: Define an inner product on V by $\alpha_i \cdot \alpha_j = A_{ij}$. Since we assumed $A_{ij} = A_{ji}$, this is a symmetric bilinear form. It is easy to check that W preserves $\cdot$. Since $\alpha_i \cdot \alpha_i = 2$, we have $\beta \cdot \beta = 2$ for any $\beta \in \Phi$, and thus, if β_1 and β_2 are proportional, then $\beta_1 = \pm\beta_2$.

7. The sign function and length

Let W be a Coxeter group, and continue to use the notations from the previous section.

Lemma. There is a homomorphism $\operatorname{sgn} : W \longrightarrow \pm 1$ given by:

$$s_i \longmapsto -1.$$

Proof. Recall $W = \langle s_i | (s_i, s_j)^{m_{ij}} = 1 \rangle$. All relations are sent to 1 if we send s_i to -1. ∎

Proof (alternate). Choose a Cartan matrix, roots, and coroots. This gives us:

$$\begin{array}{ccc} W & \longrightarrow & GL(V) \\ \downarrow{\scriptstyle \mathrm{sgn}} & & \downarrow{\scriptstyle \det} \\ \{\pm 1\} & \hookrightarrow & \mathbb{R}^* \end{array}$$ □

For $w \in W$, define $\ell(w)$ to be the minimal $\ell \in \mathbb{Z}_{\geq 0}$ such that we can write $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ for some sequence of simple reflections. Notice $\ell(w) = 0$ if and only if $w = \mathrm{Id}$. Note also that $\mathrm{sgn}(w) = (-1)^{\ell(w)}$.

We define a word $(s_{i_1}, s_{i_2}, \dots, s_{i_\ell})$ to be **reduced** if $\ell = \ell(s_{i_1} s_{i_2} \cdots s_{i_\ell})$. A (consecutive) subword of a reduced word is reduced, i.e. if $s_1 s_2 \cdots s_\ell$ is reduced, $s_i s_{i+1} \cdots s_{i+j}$ is reduced.

Lemma. For $w \in W$ and any simple generator s_i, $\ell(s_i w) = \ell(w) \pm 1$ and $\ell(w s_i) = \ell(w) \pm 1$.

Proof. We have $\ell(s_i w) \leq \ell(w) + 1$ and $\ell(w) \leq \ell(s_i w) + 1$. So, $\ell(s_i w) - \ell(w) \in \{-1, 0, 1\}$. We also have

$$(-1)^{\ell(w)} = \mathrm{sgn}(w) = -\mathrm{sgn}(s_i w) = -(-1)^{\ell(s_i w)},$$

so $\ell(w) \equiv \ell(s_i w) + 1 \mod 2$. □

We will say s_i is a **left ascent/descent of** w according to whether:

$$\ell(s_i w) = \ell(w) + 1 \text{ (left ascent), or}$$
$$\ell(s_i w) = \ell(w) - 1 \text{ (left descent).}$$

We also define

$$\ell(w s_i) = \ell(w) + 1 \text{ (right ascent), and}$$
$$\ell(w s_i) = \ell(w) - 1 \text{ (right descent).}$$

Any w other than the identity has some left descent and some right descent.

As an exercise, in S_n we have

$$\ell(w) = \#\{(i,j) | 1 \leq i < j \leq n, w(i) > w(j)\}.$$

The pairs (i,j) in this formula are called the ***inversions*** of w. So s_i is a left ascent if $w^{-1}(i) < w^{-1}(i+1)$, and is a right ascent if $w(i) < w(i+1)$.

Example. Let 231 be $2 \mapsto 1$, $3 \mapsto 2$, $1 \mapsto 3$. Then,

- s_2 is a right descent of $2\underline{31}$, because *positions* 2 and 3 are out of order, and
- s_2 is a left ascent of $\underline{23}1$, because *numbers* 2 and 3 are in order.

If you find writing 231 to denote $2 \mapsto 1$, $3 \mapsto 2$, $1 \mapsto 3$ confusing, you have my sympathies. It is, however, reasonably standard and has the advantage of working nicely with labeling the hyperplane arrangement. Specifically, $\{x_1 \leq x_2 \leq x_3\}$, $\{x_1 \leq x_3 \leq x_2\}$, $\{x_2 \leq x_3 \leq x_1\}$ are consecutive regions of the hyperplane arrangement, given by D, $s_2 D$ and $s_2 s_1 D$. We would like to abbreviate $\{x_2 \leq x_3 \leq x_1\}$ to 231, so we want $s_2 s_1$ to be 231. But $s_2(s_1(1)) = s_2(2) = 3$, $s_2(s_1(2)) = s_2(1) = 1$ and $s_2(s_1(3)) = s_2(3) = 2$, so $s_2 s_1$ is $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$.

8. A key lemma

Let W be a Coxeter group and A a Cartan matrix for W. Let α_i in V and $\alpha_i^\vee$ in $V^\vee$ with $\alpha_i^\vee, \alpha_j\rangle = A_{ij}$. We set $D = \{x^\vee \in V^\vee | \langle x^\vee, \alpha_i\rangle \geq 0\}$ and $D^0 = \{x^\vee \in V^\vee | \langle x^\vee, \alpha_i \rangle > 0\}$.

We assume $D^0 \neq \emptyset$.

Remark. Many sources assume that the α_i are linearly independent; we don't need that. However, we note that α_i and α_j are not proportional: We can't have $\alpha_i \in \mathbb{R}_{>0}\alpha_j$ as $\langle \alpha_i^\vee, \alpha_i \rangle = 2$ and $\langle \alpha_i^\vee, \alpha_j \rangle \leq 0$, and we can't have $\alpha_i \in \mathbb{R}_{<0}\alpha_j$ as that would make $D^0 = \emptyset$.

Today's goal is the following key lemma:

Lemma. For $w \in W$ and s_i a simple reflection:

- If s_i is a left ascent of w, then $\langle \quad, \alpha_i \rangle \geq 0$ on wD and
- If s_i is a left descent of w, then $\langle \quad, \alpha_i \rangle \leq 0$ on wD .

We depict the Lemma in Figure 6: We have $\langle \ , \alpha_1 \rangle \geq 0$ on the right hand half of the figure and $\langle \ , \alpha_1 \rangle \leq 0$ on the left.

Example. Let's see what this means for the symmetric group S_n. Recall that $(\sigma x)_i = x_{\sigma^{-1}(i)}$ where $\sigma \in S_n$ and $x \in \mathbb{R}^n$. So

$$x \in \sigma D \iff \sigma^{-1}x \in D \iff (\sigma^{-1}x)_i \text{ is increasing} \iff x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)}.$$

The lemma says that $(i \quad i+1)$ is a left ascent of $\sigma \iff \langle e_{i+1} - e_i, x \rangle \geq 0$ for $x \in \sigma D$. By the above equivalences, this will happen if and only if $\sigma^{-1}(i) \leq \sigma^{-1}(i+1)$.

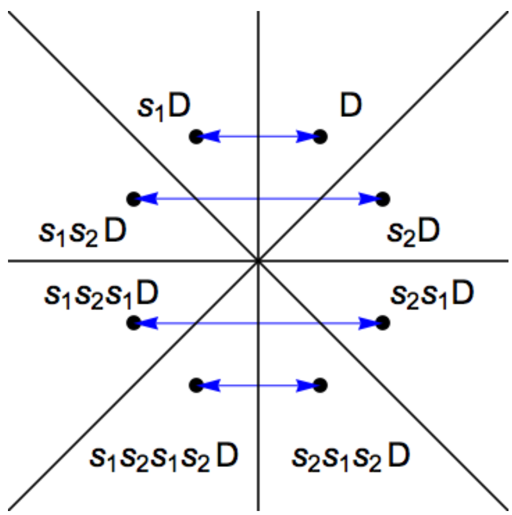


FIGURE 6. We have $\langle \ , \alpha_1 \rangle \geq 0$ on the right side of the figure and ≤ 0 on the left.

Proof. We proceed by induction on $\ell(w)$.

Base case: $\ell(w) = 0$ so $w = 1$. We want to show $\langle \quad, \alpha_i \rangle \geq 0$ on D. This is true by the definition of D.

Case I: s_i is a left descent of w so $\ell(s_i w) = \ell(w) - 1$. Inductively, $\langle x, \alpha_i \rangle \geq 0$ for $x \in s_i wD$, so $\langle s_i y, \alpha_i \rangle \geq 0$ for $y \in wD$. But $\langle s_i y, \alpha_i \rangle = \langle y, s_i \alpha_i \rangle = -\langle y, \alpha_i \rangle$ so $\langle y, \alpha_i \rangle \leq 0$ for $y \in wD$.

Case II: s_i is a left ascent of w, but $\ell(w) > 0$. This is the key case. Since $w \neq 1$, there is some left descent s_j of w. Choose a reduced word for w of the form uv where u is in the subgroup generated by s_i and s_j and u is as long as possible. Then u and v are reduced and $\ell(u) + \ell(v) = \ell(w)$.

Since we chose u maximal, v doesn't have reduced words with first letter s_i or s_j. So s_i and s_j are both left ascents of v. And $\ell(v) = \ell(w) - \ell(u) < \ell(w)$. So, inductively, $\langle \quad, \alpha_i \rangle \geq 0$ and $\langle \quad, \alpha_j \rangle \geq 0$ on vD. Geometrically, vD is somewhere in the dark gray cone of Figure 7.

We know u doesn't start with s_i since s_i is a left ascent for w, so $u = s_j s_i s_j s_i ... s_{i \text{ or } j}$ We claim that $1 \leq \ell(u) \leq m_{ij} - 1$. If $\ell(u) = 0$ then s_j is not a descent of w, contradiction. No element of $\langle s_i, s_j \rangle$ has length greater than m_{ij}. Finally, is if $\ell(u) = m_{ij}$ then u could be rewritten as $s_i s_j ...$ with m_{ij} terms, which contradicts s_i being a left ascent of w. So the claim is proven. Geometrically, we have shown that uD is one of the blue triangles in Figure 7.

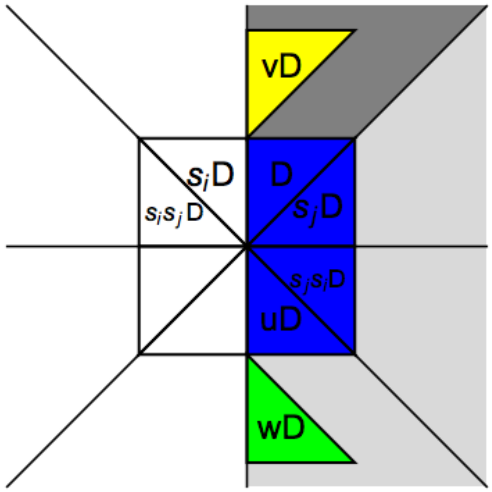


FIGURE 7. The proof of the Key Lemma

By our computations in Section 5, and the result of the previous paragraph regarding u, we know that u maps the region $\{x : \langle \alpha_i, x \rangle,\ \langle \alpha_j, x \rangle \geq 0\}$ into the half space $\{x : \langle \alpha_i, x \rangle \geq 0\}$. (This half space is shaded in Figure 7.) So $uvD = wD$ lies in $\{x : \langle \alpha_i, x \rangle \geq 0\}$, as desired. ☐

Corollary. If $w \in W$ is not 1 then $wD^0 \cap D^0 = \emptyset$.

Proof. Let s_i be a left descent of w. We have $\langle \alpha_i, \ \rangle < 0$ on wD^0 and $\langle \alpha_i, \ \rangle > 0$ on D^0. ☐

Corollary. $W \to GL(V^\vee)$ is injective. $W \to GL(V)$ is also injective.

Remark. Remember that W is defined abstractly by generators and relations. Even in type A_{n-1}, it is not obvious that these relations are enough to quotient down to the group S_n, but we have shown this must be true, because S_n is clearly the image of W in $\mathrm{GL}_n(\mathbb{R})$.

9. CONSEQUENCES OF THE KEY LEMMA

We continue to impose that $D^\circ \neq \emptyset$.

Last time, we proved the Key Lemma:

Lemma. We have $\langle \ , \alpha_i \rangle \geq 0$ on wD if s_i is a left ascent of w and we have $\langle \ , \alpha_i \rangle \leq 0$ on wD if s_i is a left descent of w.

Corollary. For any $w \in W$ other than the identity, we have $wD^\circ \cap D^\circ = \emptyset$.

Proof. Let s_i be a descent of w. Then $\alpha_i^\perp$ separates D° and wD°. ☐

Corollary. For $u, v \in W$, with $u \neq v$, we have $uD^\circ \cap vD^\circ = \emptyset$.

Proof. We have $uD^\circ \cap vD^\circ = u(D^\circ \cap u^{-1}vD^\circ)$. ☐

So all wD^o are disjoint.

Corollary. The maps $W \to GL(V)$ and $W \to GL(V^\vee)$ are injective.

We define Φ to be $\Phi = W \cdot \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset V$.

Corollary. For any $\beta \in \Phi$ and $w \in W$, the cone wD lies entirely to one side of $\beta^\perp$.

Proof. Let $\beta = u\alpha_i$. We can reduce to α_i and $u^{-1}wD$ and use the Key Lemma. □

Corollary. Each wD° is a connected component of $V \setminus \cup_{\beta\in\Phi}\beta^\perp$.

Proof. Its enough to show the claim for D°. We know D° lies entirely to one side of each $\beta^\perp$, and it is connected, so D° lies in some connected component E of $V \setminus \cup_{\beta\in\Phi}\beta^\perp$. Suppose for contradiction, there is some $x \in E \setminus D^\circ$. Since $x \in E$, for each α_i, we have $\langle x, \alpha_i\rangle \neq 0$. Moreover, since $x \notin D^\circ$, for some α_j, we have $\langle x, \alpha_i\rangle < 0$. Then $\alpha_j^\perp$ separates x from D°. □

We now know that, for any $\beta \in \Phi$ and $w \in W$, the cone wD is entirely to one side of $\beta^\perp$. In particular, for any $\beta \in \Phi$, we have either $\langle\ \ ,\beta\rangle \geq 0$ or $\langle\ \ ,\beta\rangle \leq 0$ on D. So we can write $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+ = \{\beta \in \Phi\langle D^\circ, \beta\rangle > 0\}$ are the positive roots, and $\Phi^- = \{\beta \in \Phi | \langle D^\circ, \beta\rangle < 0\}$ are the negative roots. We earlier described positive and negative roots using a particular $\rho \in D^\circ$; we now see that the choice of ρ is irrelevant.

We now describe how to see positive and negative roots in V rather than in terms of their pairing with the dual space $V^\vee$. See Figure 8 for a graphical depiction.

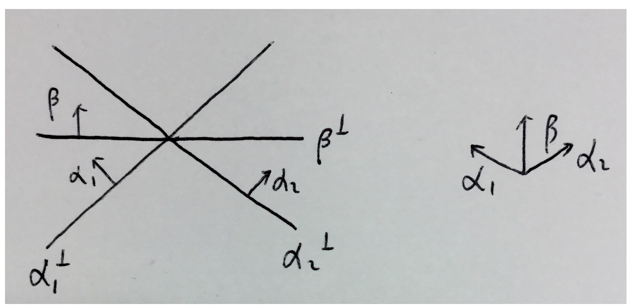


FIGURE 8. D lying on the positive side of $\beta^\perp$ is dual to β being in the positive span of the α_i

Lemma. If $\beta \in \Phi^+$, then β can be written as $\sum c_i\alpha_i$ with $c_i \geq 0$.

Proof. Suppose not. Let $R = \{\sum c_i\alpha_i, c_i \geq 0\}$, so R is a closed convex set in V and $\beta \notin R$. By the Farkas lemma, there is some $\gamma \in V^\vee$, with $\langle \gamma, \ \ \rangle \geq 0$ on R, and $\langle \gamma, \ \ \rangle < 0$ on β. But then $\gamma \in D$, and $\langle \gamma, \beta\rangle < 0$, so β is not in Φ^+ after all. □

Lemma. Choose an index i. There exists an $\theta \in D$, such that $\langle \theta, \alpha_i\rangle = 0$ and $\langle \theta, \alpha_j\rangle > 0$ for $i \neq j$.

Proof. Consider the line segment joining $\rho \in D^\circ$ and $-\alpha_i^\vee$. At ρ, all $\langle\ \ , \alpha_j\rangle > 0$, and at $-\alpha_i^\vee$, we have $\langle\ \ , \alpha_i\rangle = -2$ and $\langle\ \ , \alpha_j\rangle \geq 0$ for $i \neq j$. At the point of this segment where $\langle\ \ , \alpha_i\rangle = 0$ we must have $\langle\ \ , \alpha_j\rangle > 0$ for all $i \neq j$. □

Corollary. $\alpha_i \notin \mathrm{Span}_{\mathbb{R}^+}\{\alpha_j :\ j \neq i\}$

Proof. Let $\alpha_i = \sum_{j\neq i} c_j\alpha_j$, pair with θ in the previous lemma. □

Corollary. There is a nonempty open set of $\alpha_i^\perp$ contained in $D \cap s_iD$

Proof. Let U be a small enough neighborhood of θ in $\alpha_i^\perp$. Then all other $\langle \quad, \alpha_j\rangle > 0$ on U and $\langle \quad, \alpha_i\rangle = 0$ on U, so $U \subset D \cap \alpha_i^\perp$. But $s_iD \cap \alpha_i^\perp = s_i(D \cap \alpha_i^\perp) = D \cap \alpha_i^\perp$ since s_i fixes $\alpha_i^\perp$. □

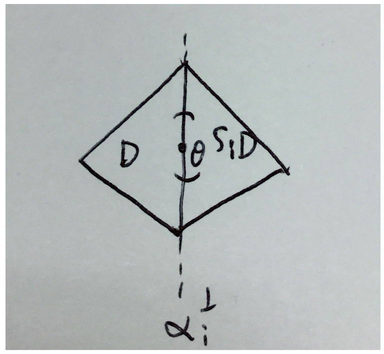


FIGURE 9. D and s_iD border along a codimension 1 wall

This shows that D and s_iD border along a codimension 1 wall, as shown in Figure 9. Thus, for any v, the cones vD and vs_iD border along a codimension 1 wall. This raises the following application, which will return next class. Consider a word $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ with product w. Set $v_k = s_{i_1}s_{i_2}\cdots s_{i_k}$. Then $v_{k-1}D$ and $v_kD = v_{k-1}s_{i_k}D$ border along a codimension one wall. So the sequence of cones $D = v_0D$, v_1D, v_2D, $\ldots$, $v_\ell D = wD$ each border along codimension one walls, as in Figure 10.

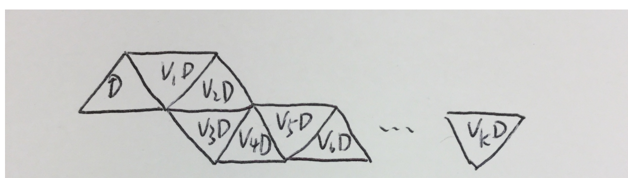


FIGURE 10. A word in W gives a walk through the chambers wD

The region $\bigcup_{w\in W} wD$ is called the ***Tits cone*** and denoted Tits(W). The above argument shows that it is connected in codimension one. We can reverse the description of the previous paragraph, using the following lemma:

Proposition. For any $\beta \in \Phi$, β not a multiple of α_i, the cones D and s_iD are on the same side of $\beta^\perp$.

Proof. If D and s_iD are on opposite sides of $\beta^\perp$, then $\langle\beta, \rangle = 0$ on $D \cap s_iD$. Then Span$(D \cap s_iD) = \alpha_i^\perp$ so $\langle\beta, \rangle$ can be 0 on it only if $\beta \in \mathbb{R}\alpha_i$. □

So the chambers bordering D are precisely the chambers s_iD. So the chambers bordering uD are us_iD. Thus, if we have a sequence of adjacent chambers D, v_1D, v_2D, $\ldots$, $v_\ell D$, we must have $v_k = v_{k-1}s_{i_k}$ for some sequence s_{i_1}, $\ldots$, s_{i_ℓ}.

We point out a useful corollary for later:

Proposition. For any $\beta \in \Phi^+$, β not a multiple of α_i, the root $s_i\beta$ is in Φ^+.

Proof. Since β is positive, we have $\langle\beta, \ \rangle > 0$ on D. So we also have $\langle\beta, \ \rangle > 0$ on s_iD, and thus $\langle s_i\beta, \ \rangle > 0$ on D. □

10. Reflections, transpositions and roots

We continue to assume $D^\circ \neq \emptyset$.

We can now clear up a few details. We defined $T = \{us_iu^{-1} : u \in W,\ i = 1, \cdots, n\}$, and we defined an element t of $\mathrm{GL}(V)$ to be a reflection if $t^2 = 1$ and t fixes a linear space of codimension 1. It is clear that the elements of T are reflections, but it is not clear that every reflection is in T. Indeed, until we knew that $W \to \mathrm{GL}(V)$ was injective, we had no hope of proving this since, if g is in the kernel of this map and $t \in T$, then gt is also a reflection. Now we can address this issue.

Proposition. Let $t \in W$ act on $V^\vee$ by a reflection. Then $t \in T$.

Proof. Let H be the fixed plane of t. Choose a word $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ for t, and define $v_k = s_{i_1}s_{i_2}\cdots s_{i_k}$ as before. Then D, v_1D, v_2D, $\dots$, $v_\ell D = tD$ is a walk through the Tits cone. Since D and tD are on opposite sides of $\mathrm{Fix}(t)$, this walk must cross it somewhere. There are two cases:

Case 1: The hyperplane H passes through the interior of some v_kD. But then $tv_kD^\circ \cap v_kD^\circ$ is nonempty, contradicting that the wD° are disjoint.

Case 2: For some index k, the hyperplane H separates $v_{k-1}D$ and v_kD. Then $tv_{k-1}D^\circ \cap v_kD^\circ$ is nonempty, so $tv_{k-1} = v_k$. We deduce that $t = v_{k-1}s_{i_k}v_{k-1}^{-1}$. □

We often describe a reflection by pointing to the hyperplane it fixes. The next Proposition justifies this:

Proposition. If t_1 and $t_2 \in T$ reflect over the same hyperplane, then $t_1 = t_2$.

Proof. Let $t_1 = u_1s_1u_1^{-1}$ and $t_2 = u_2s_2u_2^{-1}$. Consider $t_2 \cdot u_1D$. Points of u_1D^0 very near the $(u_1\alpha_1)^\perp$ wall would be mapped by t_2 into $t_1u_1D^0$. Since the interiors of the chambers are disjoint, $t_2u_1 = t_1u_1$ and hence $t_2 = t_1$. □

Remark. This need not be true when $D = \emptyset$, see the left hand side of Figure 4.

If $t = us_iu^{-1}$, then $t(x) = x - \langle \beta^\vee, x\rangle\beta$, where $\beta = u\alpha_i$ and $\beta^\vee = u\alpha_i^\vee$.

We have now shown that we have bijections between T, reflections and reflecting hyperplanes. Reflecting hyperplanes correspond to Φ modulo scaling, so these things all correspond to Φ modulo scaling. Unfortunately, we still don't have quite enough tools to address the scaling issue that we discussed in Section 6.

11. Finishing the classification of finite reflection groups

In Section 3, we classified positive definite Cartan matrices. Prior to doing that, we discussed one procedure which starts with a finite orthogonal reflection group and produces a positive definite Cartan matrix, and another which starts with a positive definite Cartan matrix and produces an orthogonal reflection group. We did not show that these procedures were inverse, so we don't know that our classification of Cartan matrices corresponds to a classification of finite reflection groups. (We did note that every finite reflection group is orthogonal, see Section 4.)

Algorithm 1: Start with W a finite orthogonal reflection group. Take

$$\Phi = \{\text{normals to hyperplanes}\}.$$

Choose ρ not in any $\beta^\perp$; this splits $\Phi = \Phi^+ \sqcup \Phi^-$. Let $\alpha_1, \dots, \alpha_k$ be simple roots in Φ^+ with reflections $s_1, \dots, s_k$. The α_i will be linearly independent, and the angle between α_i and α_j

is $\pi(1 - 1/m_{ij})$ where m_{ij} is the order of $s_i s_j$. We get a positive-definite Cartan matrix A with entries $A_{ij} = \alpha_i \cdot \alpha_j$.

Algorithm 2: Start with positive-definite Cartan matrix with entries A_{ij}. Find $\alpha_1, \dots, \alpha_k \in V$ and $\alpha_1^\vee, \dots, \alpha_k^\vee \in V^\vee$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$. Note that the α_j's will be linearly independent: indeed, if $\sum_i c_i \alpha_i = 0$, then $\langle \sum_i c_i \alpha_i^\vee, \sum_j c_j \alpha_j \rangle = 0$. However, the latter value can be expressed as $\langle \Sigma, \Sigma \rangle = c^T A c$ and A is positive definite, yielding a contradiction.

We can define an inner product on V (so $V \cong V^\vee$) and can think of $V = V^\vee$. The s_i, given by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$, generate some subgroup $W \subseteq \mathrm{GL}(V)$.

Claim 1: The output of Algorithm 2 is finite when A is positive definite.

Proof of Claim 1. Let Σ be the unit sphere in V and let $\Delta = \Sigma \cap D$. We equip Σ with a Riemmannian metric by restricting the inner product from V, so we can talk about volumes of subsets of V. All of the $w\Delta$ have disjoint interiors in Σ. So

$$\mathrm{Vol}(\Sigma) \geq \sum_{w \in W} \mathrm{Vol}(w\Delta) = |W| \,\mathrm{Vol}(\Delta)$$

where the equality is because W preserves the inner product on V and hence preserves the metric on Σ. Thus

$$|W| \leq \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\Delta)}.$$

□

Remark. In Section 12, we will show that, if W is finite, then $V = \bigcup wD$. So, in fact, $\Sigma = \bigcup w\Delta$ and $|W| = \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\Delta)}$ in this case.

We now know that it makes sense to compose the algorithms in either order. We first consider the composition

$$W_{in} \overset{Alg\ 1}{\longmapsto} A \overset{Alg\ 2}{\longmapsto} W_{out}.$$

That is, W_{in} is sent to A by Algorithm 1, and A in turn is sent to W_{out} by Algorithm 2.

Claim 2: W_{in} is isomorphic to W_{out}.

Proof of Claim 2. By definition, W_{out} is generated by $s_1^{out}, \dots, s_k^{out}$ obeying $(s_i^{out} s_j^{out})^{m_{ij}} = 1$. Meanwhile, the elements $s_1^{in}, \dots, s_k^{in}$ in W_{in} obey $(s_i^{in} s_j^{in})^{m_{ij}} = 1$. So we have a group homomorphism $W_{out} \to W_{in} \subseteq \mathrm{GL}(V)$.

We showed $W_{out} \to \mathrm{GL}(V)$ is injective, so $W_{out} \subseteq W_{in}$. The group W_{in} is generated by reflections, so it is enough to check every reflection t in W_{in} is in the image of W_{out}. Suppose that $H = \mathrm{Fix}(t)$ and that $H = \beta^\perp$ for some $\beta \in \Phi_{in}$. If $H = \beta^\perp$ for some $\beta \in \Phi_{out}$, then t is the orthogonal reflection over $\beta^\perp$ and $t \in W_{out}$. If not, H passes through wD° for some $w \in W_{out}$. Up to replacing t by $w^{-1}tw$, we may assume H passes through D°. Then

$$\beta \notin \mathrm{Span}_{\mathbb{R}^+}(\alpha_1, \dots, \alpha_k) \cup -\mathrm{Span}_{\mathbb{R}^+}(\alpha_1, \dots, \alpha_k).$$

But that means we failed to take the correct simples in Algorithm 1, a contradiction. □

Now we compose in the opposite order. Suppose we start with a positive-definite Cartan matrix A_{in} and we construct

$$A_{in} \overset{Alg\ 2}{\longmapsto} W \overset{Alg\ 1}{\longmapsto} A_{out}.$$

Under Algorithm 2 we consider the following data: $\alpha_1^{in}, \dots, \alpha_k^{in} \in V$ and a positive-definite inner product on V with $\langle s_i^{in} \rangle = W$. Under Algorithm 1 we consider the following data: $\Phi = \Phi^+ \cup \Phi^-$, where $\alpha_1^{out}, \dots, \alpha_k^{out} \in \Phi^+$ are the simples of Φ^+. We want to verify that:

- $\langle \alpha_i^{out}, \alpha_j^{out} \rangle = \langle \alpha_i^{in}, \alpha_j^{in} \rangle$.
- If ρ is such that $\langle \rho, \alpha_i^{in} \rangle > 0$, then $\{\alpha_i^{in}\} = \{\alpha_j^{out}\}$.

First of all, what if we choose ρ such that $\langle \alpha_i^{in}, \rho \rangle > 0$?

Thus Φ^+, as defined by $\langle \rho, \cdot \rangle$, will be positive combinations of the α_j^{in}'s; the latter will be the simple roots.

What about some other ρ? That ρ must lie in wD° for some $w \in W$, so $w^{-1}\rho \in D^\circ$. The positive roots for ρ are $w \cdot \{\text{positive roots for } D^\circ\} = \{w \cdot \alpha_1^{in}, \ldots, w \cdot \alpha_k^{in}\}$.

12. Inversions, reflection sequences and length

Last time, we defined $T = \{us_iu^{-1} | i = 1, \ldots, n, u \in W\}$, the set of conjugates of simple generators. We saw the bijection:

$$\begin{array}{lcl} T & \longleftrightarrow & \{\beta^\perp \ : \ \beta \in \Phi\} \\ t & \longmapsto & \mathrm{Fix}(t) \\ ws_iw^{-1} & \longleftarrow & (w\alpha_i)^\perp. \end{array}$$

Given a word $s_{i_1}s_{i_2}\cdots s_{i_\ell}$, we defined a sequence $v_k = s_{i_1}s_{i_2}\cdots s_{i_k}$ and $t_k = s_{i_1}s_{i_2}\cdots s_{i_k}\cdots s_{i_2}s_{i_1}$. The chambers $D = v_0D$, v_1D, $\ldots$, $v_\ell D$ form a walk through the Tits cone, and $v_{k-1}D$ and v_kD are separated by the hyperplane $\mathrm{Fix}(t_k)$. We call t_1, t_2, $\ldots$, t_ℓ the ***reflection sequence*** of the word $s_{i_1}s_{i_2}\cdots s_{i_\ell}$.

Suppose we have two words for w. By the definition of W as generators and relations, we can turn one word into the other by

- Inserting or deleting s_i^2 and
- Replacing $s_is_js_i\cdots s_{(i \text{ or } j)}$ by $s_js_i\cdots s_{(j \text{ or } i)}$, where these two blocks are of length m_{ij}.

This latter operation is called a ***braid move*** for $m_{ij} \geq 3$, and a ***commutation move*** for $m_{ij} = 2$. These two operations effect the reflection sequence as follows:

- Inserting s_i^2 in positions k and $k+1$ inserts two copies of the same reflection in positions $(k, k+1)$.
- Performing a braid or commutation move in positions $(k+1, k+2, \ldots, k+m_{ij})$ reverses the reflection sequence in those positions.

We thus see that, for any reflection t, the parity of the number of times t occurs in the reflection sequence is independent of the choice of reduced word. For $w \in W$, we define t to be an ***inversion*** of w if t occurs an odd number of times in reflection sequences of words for w, and write $\mathrm{inv}(w)$ for the set of inversions of w. Geometrically, we see that

$$\mathrm{inv}(w) = \{t \in T : \ \mathrm{Fix}(t) \text{ separates } D \text{ and } wD\}.$$

It is clear that $|\,\mathrm{inv}(w)| \equiv \ell(w) \bmod 2$ and $|\,\mathrm{inv}(w)| \leq \ell(w)$. We will soon show something better:

Proposition. For any $w \in W$, we have $|\,\mathrm{inv}(w)| = \ell(w)$.

Corollary. A word for w is reduced if and only if no reflection occurs twice in the reflection sequence.

Before we prove this, it is worth making a more general definition: for $x \in V^\vee - \bigcup_{\beta\in\Phi} \beta^\perp$, set

$$I(x) = \{t \in T \ : \ \mathrm{Fix}(t) \text{ separates } D \text{ and } x\}.$$

So $I(x) = \mathrm{inv}(w)$ for $x \in wD^\circ$.

Proposition. For x as above and s_i a simple reflection, we have $I(s_i x) = s_i I(x) \triangle \{s_i\}$, where $\triangle$ is symmetric difference.

Proof. It is clear that x and $s_i x$ are on opposite sides of $\alpha_i^\perp$, so $s_i \in I(x)$ if and only if $s_i \notin I(s_i x)$. For any reflection t other than s_i, let β be a corresponding positive root. Then $s_i\beta$ is also a positive root. We have $\langle \beta, x\rangle > 0$ if and only if $\langle s_i\beta, s_i x\rangle > 0$. □

We now show that $|\operatorname{inv}(w)| = \ell(w)$, as claimed above, and something more.

Proposition. Let $x \in V^\vee - \bigcup_{\beta\in\Phi} \beta^\perp$ and suppose that $I(x)$ is finite. Then $x \in wD$ for some w with $\ell(w) = |I(x)|$.

Proof. Our proof is by induction on $|I(x)|$. If $|I(x)| = \emptyset$, then $\langle \alpha_i,\ x\rangle > 0$ for every simple root α_i, so $x \in D$ and we take $w = e$.

Now, suppose that $I(x)| \neq \emptyset$ so $x \notin D$. Then some $\langle \alpha_i, x\rangle < 0$ and $s_i \in I(x)$. Then $|I(s_i x)| = |I(x)| - 1$ so, by induction, we have $s_i x \in w'D$ for some w' with $\ell(w') = |I(x)| - 1$. Then $x \in s_i w'D$. Moreover, since $\langle \alpha_i, x\rangle > 0$ on $w'D$, we know that s_i is a left-ascent of w', so $\ell(s_i w') = \ell(w') + 1$. Taking $w = s_i w'$, we are done. □

We have now established that the number of inversions of w is the length of w. There are also some other useful corollaries of this result:

Corollary. Let $x \in V^\vee - \bigcup_{\beta\in\Phi} \beta^\perp$. Then $x \in \text{Tits}$ if and only if $I(x)$ is finite.

Corollary. Let $x \in V^\vee$. Then $x \in \text{Tits}$ if and only if $\{t \in T : \langle \beta_t, x\rangle < 0\}$ is finite.

From this we can deduce:

Corollary. We have $\text{Tits} = V^\vee$ if and only if W is finite, if and only if $-D \subseteq \text{Tits}$.

If W is finite, then there must be some w_0 with $w_0 D = -D$. We have $\operatorname{inv}(w_0) = T$, so w_0 is the longest element of W. The element w_0 is called the ***long word*** or the ***longest word*** of W.

13. Root sequences

Last time, given a word $s_{i_1} s_{i_2} \cdots s_{i_\ell}$, we defined a sequence $v_k = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $t_k = s_{i_1} s_{i_2} \cdots s_{i_k} \cdots s_{i_2} s_{i_1}$. Put $\beta_k = s_{i_1} s_{i_2} \cdots s_{k-1} \alpha_{i_k}$. So $\beta_k^\perp$ is the fixed space of t_k. We call β_1, ..., β_ℓ the ***root sequence***.

When we walk from $t_{k-1}D$ to $t_k D$, we cross from the side where $\langle \beta_k,\ \rangle$ is positive to where it is negative. This has the following easy consequences:

Proposition. Fix a reflecting hyperplane H. Then vectors normal to H occur in the reflection sequence with alternately positive and negative sign, starting with positive.

Corollary. The word $s_{i_1} s_{i_2} \cdots s_{i_k}$ is reduced if and only if all the β_k are positive.

This is a practical condition to use to test whether a word is reduced.

Finally, we have the tools to address the annoying issue of proportional positive roots. As we noted way back in Section 5, if m_{ij} is odd and $A_{ij} \neq A_{ji}$, then $(s_i s_j)^{(m-1)/2}\alpha_i = \sqrt{\frac{A_{ji}}{A_{ij}}}\alpha_j$. We therefore make the hypothesis that, $\boxed{\text{if } m_{ij} \text{ is odd, then } A_{ij} = A_{ji}}$.

With this hypothesis in place, we analyze the effect of changing the word $s_{i_1} \cdots s_{i_k}$ on the reflection sequence.

Proposition. Inserting s_i^2 in positions k and $k+1$ inserts a β and a $-\beta$ in position $(k, k+1)$.

Proof. Put $u = s_{i_1} \cdots s_{i_{k-1}}$. We are considering the roots $u\alpha_i$ and $us_i\alpha_i$. Clearly, $us_i\alpha_i = u(-\alpha_i) = -u\alpha_i$. □

Proposition. With the hypothesis that, if m_{ij} is odd, then $A_{ij} = A_{ji}$, performing a braid or commutation move on s_i and s_j in positions $(k+1, k+2, \ldots, k+m_{ij})$ reverses the root sequence in those positions.

Proof. Put $u = s_{i_1} \cdots s_{i_k}$. We want to consider, on the one hand, $(u\alpha_i, us_i\alpha_j, us_is_j\alpha_i, \cdots)$ and, on the other hand, $(u\alpha_j, us_j\alpha_i, us_js_i\alpha_j, \cdots)$. We observed in Section 5 that the sequences $(\alpha_i, s_i\alpha_j, s_is_j\alpha_i, \cdots)$ and $(\alpha_j, s_j\alpha_i, s_js_i\alpha_j, \cdots)$ are reversals of each other, and acting on everything by u doesn't change that. □

Corollary. Consider any root $\beta \in \Phi$. The elements in the reflection sequence of the form $\pm\beta$ appear alternately, starting with the positive root. If β occurs an odd number of times in the reflection sequence for one word, then it also occurs an odd number of times in the reflection sequence for any other word.

We are finally ready to address the nuisance issue. The importance of this result is much smaller than the amount of time it has taken up; it just keeps coming up when we want to state things carefully:

Theorem. Suppose that if m_{ij} is odd, then $A_{ij} = A_{ji}$, and that $D^\circ \neq \emptyset$. Let $u\alpha_i$ and $v\alpha_j$ be proportional roots. Then $u\alpha_i = \pm v\alpha_j$.

Proof. Since $u\alpha_i$ and $v\alpha_j$ are proportional, us_iu^{-1} and vs_jv^{-1} are reflections over the same hyperplane and thus $us_iu^{-1} = vs_jv^{-1}$. We rearrange this equation as $(v^{-1}u)s_i = s_j(v^{-1}u)$, and put $w = v^{-1}u$.

We first make some preliminary clean up: The hypotheses and desired conclusion are unchanged by replacing u by us_i and/or replacing v by vs_j. These have the effect of replacing w by ws_i, s_jw or s_jws_i. We may therefore assume that s_i is a right ascent, and s_j a left ascent, of w. (This reduction is the place where the $\pm$ comes from. Once we have made this reduction, we can show the sign is $+$.)

Choose a reduced word $s_{k_1} \cdots s_{k_r}$ for w. By our reduction in the previous paragraph, the words $s_{k_1} \cdots s_{k_r}s_i$ and $s_js_{k_1} \cdots s_{k_r}$ are reduced as well. So there is only one root in the reflection sequence of $s_js_{k_1} \cdots s_{k_r}$ which is proportional to α_j, namely the α_j in the first position. But then α_j must occur in the reflection sequence of $s_{k_1} \cdots s_{k_r}s_i$ and, since that word is reduced as well, it must occur in the last position. We deduce that $\alpha_j = s_{k_1} \cdots s_{k_r}\alpha_i = w\alpha_i$. So $\alpha_j = w\alpha_i = v^{-1}u\alpha_i$, and we have $v\alpha_i = u\alpha_j$ as desired. □

14. Parabolic Subgroups

Let I be a subset of the vertices of the Dynkin diagram. The ***standard parabolic subgroup*** W_I is the subgroup of W generated by s_i for $i \in I$.

We choose a Cartan matrix A, roots and coroots α_i and $\alpha_i^\vee$ as usual, such that D° is nonempty. Let A_I be the submatrix of A with rows and columns indexed by I. Then $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ pair by A_I. Setting $D_I^\circ = \{x \in V^\vee \ : \ \langle x, \alpha_i\rangle > 0 \ i \in I\}$, we have $D_I^\circ \supseteq D^\circ$ and in particular is nonempty. Thus, all of our results apply to $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$, and we get to deduce that the group W_I is the Coxeter group for the generators $\{s_i\}_{i\in I}$, with relations coming from m_{ij} with i, $j \in I$.

Put $L = \{x \in V^\vee : \langle x, \alpha_i\rangle = 0 \ i \in I\}$. So we have built a hyperplane arrangement for W_I in $V^\vee$ where all the hyperplanes contain the linear space L. The Tits cone for W_I contains the Tits cone of W, so every cone wD° for $w \in W$ is contained in $w_I D_I^\circ$ for a unique $w_I \in W_I$. We can characterize w_I alternately by saying that the inversions of w_I are $\mathrm{inv}(w) \cap W_I$.

If we write w as $w_I{}^I w$, then we have with $\ell(w) = \ell(w_I) + \ell({}^I w)$. We note that we have $w_I = e$, and equivalently w is of the form ${}^I(\ )$, if and only if s_i is a left ascent of w for $i \in I$.

We have previously seen that W has trivial stabilizer on the points in D°. We are now ready to describe what happens on the boundary. Let $x \in D$ and let I be the set of indices i for which $\langle x, \alpha_i\rangle = 0$.

Theorem. With the above notation, the stabilizer of x is W_I. More strongly, $wx \in D$, if and only if $w \in W_I$.

Proof. Clearly, W_I stabilizes x and also, clearly, if w stabilizes x, then $wx \in D$. Thus, the nontrivial fact is that, if $wx \in D$, then $w \in W_I$.

So, let $wx \in D$ or, equivalently, $x \in w^{-1}D$. We abbreviate $(w^{-1})_I$ as u^{-1} and ${}^I(w^{-1})$ as v^{-1}, so $w = vu$. Since $u \in W_I$, we have $wx = vx$. We may thus replace w by v. Having made this replacement, our goal will be to show that $v = e$.

If $v \neq e$, then it has a right descent s_j, and so s_j is a left descent of v^{-1}. Since v^{-1} is of the form ${}^I(\ )$, we know that s_i is a left ascent of v^{-1} for all $i \in I$, so we must have $j \notin I$.

Since s_j is a left descent of v^{-1}, by the key lemma, $\langle \alpha_j, y\rangle \leq 0$ for $y \in v^{-1}D$, and in particular $\langle \alpha_j, x\rangle \leq 0$. But also $x \in D$, so $\langle \alpha_j, x\rangle \geq 0$. We conclude that $\langle \alpha_j, x\rangle = 0$, and thus j is in I after all, a contradiction. □

Corollary. For each y in the Tits cone, there is precisely one point in the orbit of y that lies in D.

If $w^{-1}y \in D$, then use $w^{-1}y$ to define I as above.

Corollary. With the above notation, the stabilizer of y is wW_Iw^{-1}.

It is interesting to consider the classification of regular polytopes, also known as platonic solids, from this perspective. A ***regular polytope*** is a polytope in $\mathbb{R}^n$ whose symmetry group acts transitively on chains (vertex, edge, two face, ..., facet). They correspond to the ways to put edges on the path of length n to make a finite Coxeter diagram. Specifically, the symmetry group of a regular polytope is always a finite reflection group. The centers of the k-dimensional faces have stabilizers conjugate to $W_{\{1,2,\ldots,k,k+2,\ldots,n\}}$. In Table 2, the vertices of the path are numbered from left to right. There are three infinite families in this table: The simplex is the convex hull of the standard basis vectors e_j in $\mathbb{R}^n$; the hypercube is the convex hull of the 2^n vectors $\pm e_1 \pm e_2 \cdots \pm e_n$, and the cross polytope is the convex hull of the $2n$ vectors $\pm e_i$.

Remark. Be warned that, while "simplex" and"hypertetrahedon" are synonyms, "hypersimplex" means something else. The hypersimplex is the convex hull of the $\binom{n}{k}$ vectors given as the S_n orbit of $e_1 + e_2 + \cdots + e_k \in \mathbb{R}^n$. This has symmetry group S_n (or $S_n \times \{\pm 1\}$ when $k = n/2$), but is not a regular polytope except in the cases $k = 1$, $k = n - 1$ and $(k, n) = (2, 4)$.

A_n	$\bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \text{---} \bullet$	simplex (also known as hypertetrahedron)
B_n	$\bullet \overset{4}{\text{---}} \bullet \text{---} \cdots \text{---} \bullet \text{---} \bullet$	hypercube
B_n	$\bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \overset{4}{\text{---}} \bullet$	cross polytope (also known as hyperoctahedron)
F_4	$\bullet \text{---} \bullet \overset{4}{\text{---}} \bullet \text{---} \bullet$	24-cell
H_3	$\bullet \overset{5}{\text{---}} \bullet \text{---} \bullet$	dodecahedron
H_3	$\bullet \text{---} \bullet \overset{5}{\text{---}} \bullet$	icosahedron
H_4	$\bullet \overset{5}{\text{---}} \bullet \text{---} \bullet \text{---} \bullet$	120-cell
H_4	$\bullet \text{---} \bullet \text{---} \bullet \overset{5}{\text{---}} \bullet$	600-cell
$I_2(m)$	$\bullet \overset{m}{\text{---}} \bullet$	regular m-gon

TABLE 2. The regular polytopes and their symmetry groups

15. Crystallographic groups

We have built our Coxeter groups to act on a vector space V. We now discuss when W fixes some lattice in V. We start by recalling the basic linear algebra of lattices.

Definition. Let V be a finite dimensional vector space. Then, $\Lambda \subseteq V$ is called a **lattice** if it is a discrete additive subgroup of V which spans V as a vector space.

Theorem. Let Λ be a lattice in V. Then we can choose a basis of V as a real vector space which is also a basis of Λ as a free $\mathbb{Z}$-module.

Proof sketch: By induction on n; the base case $n = 0$ is vacuously true.

Now, suppose that $n > 0$. Choose a nonzero vector $w_n \in \Lambda$ such that $\frac{1}{k} w_n \notin \Lambda$ for integers $k > 1$. We can always do this by discreteness. Then $\Lambda/\mathbb{Z}w_n$ injects in $V/\mathbb{R}w_n$, call the image $\overline{\Lambda}$. Then $\overline{\Lambda}$ is again a lattice (details left to the reader), so choose a basis $\overline{w}_1, \ldots, \overline{w}_{n-1}$ of $\overline{\Lambda}$ as required. Lift $\overline{w}_j$ to $w_j \in \Lambda$. Then $w_1, \ldots, w_{n-1}, w_n$ has the required properties (details left to reader). □

If $\Lambda_1 \subset \Lambda_2$ are two lattices, then the lattices between Λ_1 and Λ_2 are in bijection with the subgroups of Λ_2/Λ_1.

Definition. For a lattice Λ in V, the **dual lattice** $\Lambda^\vee \subseteq V^\vee$ is

$$\Lambda^\vee := \{x \in V^\vee : \langle x, \beta \rangle \in \mathbb{Z} \text{ for all } \beta \in \Lambda\}.$$

Note that $\Lambda_1 \subseteq \Lambda_2$ if and only if $\Lambda_1^\vee \supseteq \Lambda_2^\vee$.

Definition. For W acting on V, we say that W **preserves** Λ iff $w(\Lambda) = \Lambda$ for all $w \in W$.

Note that if W preserves Λ then W also preserves $\Lambda^\vee$ (acting via the dual action).

We will show that, roughly, if W is a Coxeter group, W preserves some lattice in V iff $A_{ij} \in \mathbb{Z}$. More specifically, we have the following two main theorems:

Theorem. If all $A_{ij} \in \mathbb{Z}$ and $\mathbb{Z}\langle \alpha_i \rangle$ and $\mathbb{Z}\langle \alpha_j^\vee \rangle$ are lattices, then W preserves them and preserves any lattice Λ between $\mathbb{Z}\langle \alpha_i \rangle$ and $\mathbb{Z}\langle \alpha_j^\vee \rangle^\vee$.

Proof. Recall $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$ so if all $A_{ij} \in \mathbb{Z}$, then s_i acts on $\mathbb{Z}\alpha_j$ by an integer matrix and W preserves $\mathbb{Z}\alpha_j$. Likewise W preserves $\left(\mathbb{Z}\alpha_j^\vee\right)^\vee$.

To show W preserves any Λ between these, it is enough to show that W acts trivially on $(\mathbb{Z}\alpha_i^\vee)^\vee / \mathbb{Z}\alpha_i$. Let $v \in (\mathbb{Z}\alpha_i^\vee)^\vee$. Then, $s_i(v) = v - \langle \alpha_i^\vee, v\rangle \alpha_i \equiv v \bmod \mathbb{Z}\alpha_i$. □

Theorem. If W preserves a lattice in V, then we can choose positive real scalars c_i so that, after replacing the α_i by $c_i\alpha_i$ and the $\alpha_i^\vee$ by $c_i^{-1}\alpha_i^\vee$, we have $\mathbb{Z}\langle\alpha_i\rangle \subseteq \Lambda \subseteq (\mathbb{Z}\langle\alpha_i^\vee\rangle)^\vee$ and $\mathbb{Z}\langle\alpha_i^\vee\rangle \subseteq \Lambda^\vee \subseteq (\mathbb{Z}\langle\alpha_i\rangle)^\vee$. With this rescaling, the A_{ij} are integers.

Proof. Let W preserve Λ. Then, we claim that $\mathbb{R}\alpha_i \cap \Lambda \neq (0)$. To see this, note that Λ spans V, so it contains $\lambda \notin (\alpha_i^\vee)^\perp$. Then, $s_i(\lambda) = \lambda + c\alpha_i$ for some $c \neq 0$, so $c\alpha_i \in \Lambda$. Thus, the claim holds.

Since the lattice is discrete, $\mathbb{R}\alpha_i \cap \Lambda = \mathbb{Z}\cdot(c\alpha_i)$ for some $c \in \mathbb{R}_{\neq 0}$. Rescale so $c = 1$. Now, $\mathbb{Z}\langle\alpha_i\rangle_{i=1,\ldots,n} \subseteq \Lambda$. Then, $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i \in \Lambda$, so $A_{ij}\alpha_i \in \Lambda$ and $A_{ij} \in \mathbb{Z}$.

Finally, for any $\lambda \in \Lambda$, $s_i(\lambda) = \lambda - \langle\alpha_i^\vee, \lambda\rangle\alpha_i \in \Lambda$ so $\langle\alpha_i^\vee, \lambda\rangle\alpha_i \in \Lambda$, which means that $\langle\alpha_i^\vee, \lambda\rangle \in \mathbb{Z}$. Thus, we have shown that $\Lambda \subseteq (\mathbb{Z}\langle\alpha^\vee\rangle)^\vee$. □

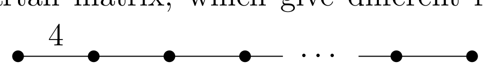
We use the following terminology, inspired by ideas from Lie groups: $\mathbb{Z}\langle\alpha\rangle$ is the ***root lattice*** and $\mathbb{Z}\langle\alpha^\vee\rangle^\vee$ is the ***weight lattice***. The corresponding lattices in $V^\vee$ are the ***coroot lattice*** and the ***coweight lattice***.

Suppose A is a crystallographic Cartan matrix with α_i a basis for V. Let $\omega_i^\vee$ be the dual basis to α_i (in other words, $\langle\omega_i^\vee, \alpha_j\rangle = \delta_{ij}$). From the equations $\langle\alpha_i^\vee, \alpha_j\rangle = A_{ij}$, we deduce that $\alpha_i^\vee = \sum A_{ij}\omega_j^\vee$. So the transition matrix from the $\omega_j^\vee$ to the $\alpha_i^\vee$ is given by A_{ij}. The $\omega_i^\vee$ are a basis for $ZZ\langle\alpha^\vee\rangle^\vee$ and the $\alpha_j^\vee$ are a basis for $\mathbb{Z}\langle\alpha_j^\vee\rangle$. So $\mathbb{Z}\langle\alpha^\vee\rangle^\vee/\mathbb{Z}\langle\alpha\rangle^\vee$ is isomorphic to the cokernel of A. Here are the values of the cokernel for the finite connected crystallographic groups.

	A_n	B_n	C_n	$D_n,\ n$ even	$D_n,\ n$ odd	E_6	E_7	E_8	F_4	G_2
$\mathbb{Z}\langle\alpha_i\rangle^\vee/\mathbb{Z}\langle\alpha_i^\vee\rangle$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1	1	1

We conclude by discussing examples. We begin with the rank two examples. In Figures 11, 12 and 13, the origin is at the center. The black circles are the roots, and thus the generators of the root lattice $\mathbb{Z}\langle\alpha\rangle$. The lines are where pairing with some $\alpha_i^\vee$ gives an integer and the open circles are the weight lattice $\mathbb{Z}\langle\alpha^\vee\rangle^\vee$.

Observe that the index of the root lattice in the weight lattice is 3, 2 and 1 in types A_2, B_2 and G_2 respectively, in accord with the above table.

We return to the issue of why B_n and C_n have different names. They correspond to different choices of crystallographic Cartan matrix, which give different root systems. Recall that $B_n = C_n$ has Coxeter diagram . Thus, we have $A_{12}A_{21} = 2$, we have $A_{i,i+1}A_{i+1,i} = 1$ for $i > 1$ and $A_{ij} = 0$ for $|i-j| > 1$. If we also impose that the A_{ij} are integers then we must have $A_{i,i+1} = A_{i+1,i} = -1$ for $i > 1$, and we either have $A_{12} = -2$, $A_{21} = -1$ or vice versa. The former case is B_n and the latter is C_n.

We can realize B_n in coordinates as $\Phi = \{\pm e_k, \pm e_i \pm e_j\}$ and $\Phi^\vee = \{\pm 2e_k, \pm e_i \pm e_j\}$. In C_n, the roles of Φ and $\Phi^\vee$ are reversed. For B_n, we have $\mathbb{Z}\langle\alpha_i\rangle = \mathbb{Z}^n$, while $\mathbb{Z}\langle\alpha^\vee\rangle^\vee = \mathbb{Z}^n + \mathbb{Z}(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. For C_n, we have $\mathbb{Z}\langle\alpha\rangle = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \sum a_i \equiv 0 (\mathrm{mod}\ 2)\}$, while $\mathbb{Z}\langle\alpha^\vee\rangle^\vee = \mathbb{Z}^n$.

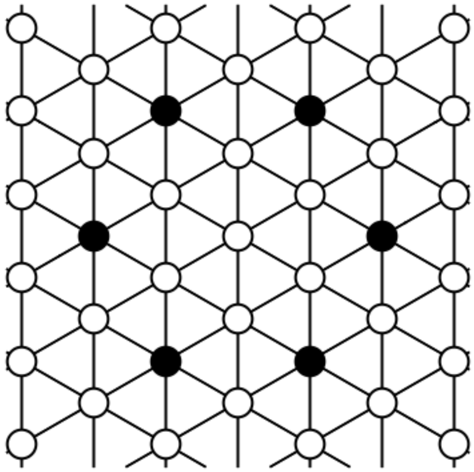


Figure 11. The A_2 roots and weight lattice

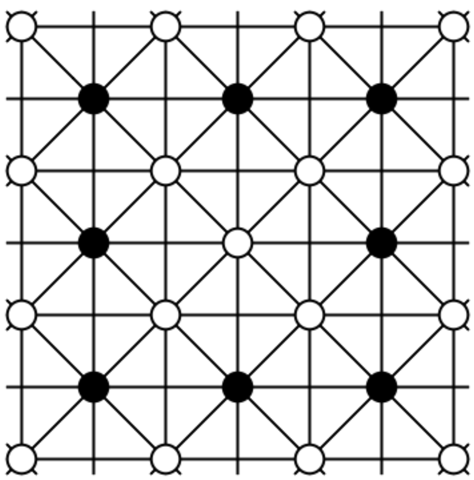


Figure 12. The $B_2 = C_2$ roots and weight lattice

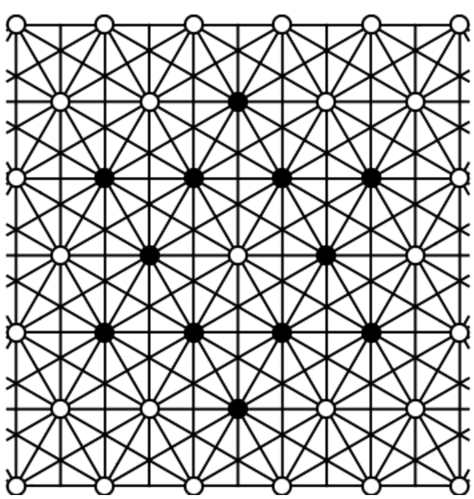


Figure 13. The G_2 roots and weight lattice

16. Affine symmetries and affine reflection groups

Let V_0 be a real vector space. Let V_1 be a principal homogenous space (also called affine space) for V_0, meaning that V_0 acts freely and transitively on V_1. Concretely, if we choose a base point z in V_1, then $\vec{v} \mapsto z + \vec{v}$ identifies V_0 with V_1. The ***affine symmetry group*** $\mathrm{AGL}(V_1)$ of V_1 can be identified with $\mathrm{GL}(V_0) \ltimes V_0$. We note that the maps $V_0 \to \mathrm{AGL}(V_1)$ and $\mathrm{AGL}(V_1) \to \mathrm{GL}(V_0)$ are completely natural, whereas the choice of right splitting $\mathrm{AGL}(V_1) \leftarrow \mathrm{GL}(V_0)$ corresponds to choosing a base point in V_1 to be fixed by the image of $\mathrm{GL}(V_0)$.

If V_0 has an inner product, then the ***Euclidean symmetry group*** $\mathrm{Euc}(V_1)$ is the subgroup $O(V_0) \ltimes V_0$ of $\mathrm{AGL}(V_1)$. Classical Euclidean geometry studies properties of $\mathbb{R}^2$ and $\mathbb{R}^3$ which are preserved by the Euclidean symmetry group.

From this perspective, reflection over an affine hyperplane is an element of $\mathrm{GL}(V_0) \ltimes V_0$ of the form $(\sigma, \vec{v})$ where σ is a reflection and $\sigma(\vec{v}) = -\vec{v}$. [1] This reflection is ***orthogonal*** if σ is orthogonal.

There is a standard trick to embed all these ideas into ordinary linear algebra in one more dimension. Let $V = V_0 \oplus \mathbb{R}$. Then $\mathrm{GL}(V_0) \ltimes V_0$ embeds in $\mathrm{GL}(V)$ as $(g, \vec{v}) \mapsto \left[\begin{smallmatrix} g & \vec{v} \\ 0 & 1 \end{smallmatrix}\right]$. We can think of V_1 as the set of pairs $(x, 1) \in V_0 \oplus \mathbb{R}$.

Taking account of the inner product is slightly trickier. Let's assume that $\cdot_0$ is a nondegenerate symmetric bilinear form on V_0, so we can (and will) identify V_0 and $V_0^\vee$. Let $V = V_0 \oplus \mathbb{R}$ with degenerate bilinear form $(\vec{x}, a) \cdot (\vec{y}, b) = \vec{x} \cdot_0 \vec{y}$. The full symmetry group of this bilinear form is $\left[\begin{smallmatrix} O(V_0) & 0 \\ V_0^\vee & \mathbb{R}_{\neq 0} \end{smallmatrix}\right]$. The corresponding dual action on $V^\vee$ is $\left[\begin{smallmatrix} O(V_0) & V_0 \\ 0 & \mathbb{R}_{\neq 0} \end{smallmatrix}\right]$. The Euclidean symmetry group thus embeds in $\mathrm{GL}(V^\vee)$ as $\left[\begin{smallmatrix} O(V_0) & V_0 \\ 0 & \mathbb{R}_{\neq 0} \end{smallmatrix}\right]$. If we put $\delta = (\vec{0}, 1) \in V$, then we can identify V_1 with the affine hyperplane $\{x : \langle \delta, x \rangle = 1\}$ in $V^\vee$.

This trick is reversible. Let V be a real vector space equipped with a symmetric bilinear form $\cdot$ that has 1-dimensional kernel $\mathbb{R}\delta$. Put $V_0 = V/\mathbb{R}\delta$. Then $\cdot$ descends to a nondegenerate symmetric bilinear form $\cdot_0$ on V_0. Put $V_1 = \{x : \langle \delta, x \rangle = 1\}$ in $V^\vee$. Then V_1 is a principal homogenous space for V_0. The group of symmetries of V preserving $\cdot$ and δ is the Euclidean symmetry group of V_1.

Suppose that $G_0 \subset \mathrm{GL}(V_0)$ is a reflection group preserving a lattice Λ, and consider the group $G = G_0 \ltimes \Lambda$ inside the affine symmetry group. Let $R \subseteq \Lambda$ be the lattice generated by those $\vec{v} \in \Lambda$ which are (-1)-eigenvectors of some reflection in G_0. Clearly, R is G_0-invariant.

We claim that the subgroup of $\mathrm{GL}(V_0) \ltimes V_0$ generated by affine reflections (σ, v) with $\sigma \in G_0$ and $\vec{v} \in \Lambda$ is precisely $G_0 \ltimes R$. Indeed,

$$(\sigma_1, v_1)(\sigma_2, v_2) \cdots (\sigma_k, v_k) = \left(\sigma_1 \sigma_2 \cdots \sigma_k, \ \sum_{j=1}^{k} \sigma_1 \sigma_2 \cdots \sigma_{j-1} v_j \right)$$

which is plainly in $G_0 \ltimes R$. Conversely, pairs $(\sigma, 0)$ generate G_0 (since G_0 is a reflection group) and $(\sigma, v)(\sigma, 0) = (e, v)$ for any reflection σ and (-1)-eigenvector $\vec{v}$ of σ. Combining these, we see that the group generated by the (σ, v) contains G_0 and contains R, so it contains $G_0 \ltimes R$.

Thus, in particular, if W_0 is a crystallographic Coxeter group, then $W_0 \ltimes \mathbb{Z}\langle \alpha \rangle$ and $W_0 \ltimes \mathbb{Z}\langle \alpha^\vee \rangle$ are reflection groups. The reflecting hyperplanes of $W_0 \ltimes \mathbb{Z}\langle \alpha \rangle$ are of the form $\{x \in V_0 : \langle \beta^\vee, x \rangle = k\}$ for $\beta^\vee \in \Phi^\vee$ and $k \in \mathbb{Z}$.

[1] If we don't have $\sigma(\vec{v}) = -\vec{v}$, then (σ, v) is a "glide reflection", and has infinite order.

17. Overview of discrete Euclidean reflection groups

Earlier, we classified finite orthogonal reflection groups. Classifying finite Euclidean reflection groups doesn't bring any new examples: If we have only finitely reflecting hyperplanes, then the hyperplane arrangement has finitely many vertices, and the group must preserve the center of mass of these vertices. So all finite Euclidean reflection groups fix a point and are thus just finite groups.

Rather, the natural thing to look at in the Euclidean case is ***discrete reflection groups***. Let $G \subset \mathrm{Euc}(V_1)$ be a group generated by reflections. Consider the corresponding hyperplane arrangement $\bigcup \mathrm{Fix}(t)$. We'll say that G is ***discrete*** if any bounded region of V_1 only meets finitely many reflecting hyperplanes $\mathrm{Fix}(t)$.

Before we state our main results, some notation. An $n \times n$ Cartan matrix A is called ***crystallographic*** if all A_{ij} are integers. The ***Coxeter diagram*** of A is the graph Γ with vertices 1, 2, …, n and an edge (i, j) if $A_{ij} \neq 0$. Finally, we make the following standard abuse of notation: Suppose that A_{ij} is a Coxeter matrix. We will say that A is ***positive (semi)-definite*** if there are positive scalars d_i such that $d_i A_{ij} = d_j A_{ji}$ and the symmetric matrix $\left[\, d_i A_{ij} \,\right]$ is positive (semi)-definite.

We now state our main results:

Theorem. All the discrete Euclidean reflection groups are Coxeter groups. More specifically, they correspond to positive semi-definite Cartan matrices module the equivalence relation $\left[\, A_{ij} \,\right] \sim \left[\, \frac{c_i}{c_j} A_{ij} \,\right]$.

Since the discrete Euclidean reflection groups are Coxeter groups, they are products of the Coxeter groups coming from the connected components of the Coxeter diagram Γ. So it is enough to understand the connected positive semi-definite Cartan matrices.

Theorem. Let $\tilde{A}$ be a connected positive semi-definite Cartan matrix. Then $\tilde{A}$ has signature either $+ + + \cdots +$ or $+ + + \cdots + 0$.

In the first case, we get a finite Coxeter group, which we have already classified. We define a Cartan matrix to be ***affine*** if it is connected and positive semi-definite. We define a Coxeter group to be ***affine*** if it comes from an affine Cartan matrix. The affine groups also have a beautiful classification:

Theorem. The affine Coxeter groups are in bijection with the connected crystallographic *positive definite* Cartan matrices, *not* modulo the relation $\left[\, A_{ij} \,\right] \sim \left[\, \frac{c_i}{c_j} A_{ij} \,\right]$. More specifically, if $\tilde{W}$ is the affine group and $(W, \Phi, \Phi^\vee)$ is the corresponding finite crystallographic group, root system and co-root system, then $\tilde{W} \cong W \ltimes \mathbb{Z}\langle \alpha_i^\vee \rangle$. The subgroup W of $\tilde{W}$ is a parabolic subgroup on one fewer generator than $\tilde{W}$.

More concretely, let $V_0^\vee$ be the reflection representation of W; then $\tilde{W}$ is the subgroup of $\mathrm{Euc}(V_0^\vee)$ generated by orthogonal reflections over the hyperplanes $\{x \in V_0^\vee : \langle x, \beta \rangle = k\}$ for $\beta \in \Phi$ and $k \in \mathbb{Z}$. The subgroup W is generated by the reflections in $\{x \in V_0^\vee : \langle x, \beta \rangle = 0\}$.

Stated dually, we can take the root systems in $V_0 \oplus \mathbb{R}$ to be given by $\{(\beta, k) : \beta \in \Phi,\ k \in \mathbb{Z}\}$ and $\{(\beta^\vee, 0) : \beta^\vee \in \Phi^\vee\}$, where $V_0 \oplus \mathbb{R}$ has degenerate bilinear form as described in the previous section.

Example. Consider A_{n-1} acting on $\mathbb{R}^n$ in the standard way. Then $\tilde{A}_{n-1}$ acts by reflections over hyperplanes of the form $\{x_i - x_j = k\}$. Concretely, this is

$$(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_j + k, \ldots, x_i - k, \ldots, x_n).$$

The simple reflections are the s_i, which generate a copy of the finite group S_n, and the additional reflection

$$s_0(x_1, x_2, \dots, x_{n-1}, x_n) = (x_n - 1, x_2, \dots, x_{n-1}, x_1 + 1).$$

There is a natural way to think of $\tilde{A}_{n-1}$ as permutations of the integers: Let V_1 be the affine space of bi-infinite sequences x_i obeying $x_{i+n} = x_i + 1$. Each $(x_1, \dots, x_n)$ extends uniquely to such a bi-infinite sequence, and the s_i can be thought of as the permutations $\cdots (i-n, i+1-n)(i, i+1)(i+n, i+1+n)(i+2n, i+1+2n) \cdots$ of $\mathbb{Z}$.

Example. The group $\tilde{C}_n$ acts on $\mathbb{R}^n$ by reflections over the hyperplanes $x_i \pm x_j = k$ and $2x_i = k$, for $1 \leq i < j \leq n$ and $k \in \mathbb{Z}$. Simple hyperplanes can be taken to be the simple hyperplanes for C_n, namely $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, together with the additional inequality $x_n \leq 1/2$. This can also be thought of as the group of permutations of infinite sequences: Consider bi-infinite sequences x_i obeying $x_{i+2n} = x_i + 1$ and $x_{1-j} = -x_j$. So every $(x_1, x_2, \dots, x_n)$ extends uniquely to such a sequence $(\dots, -x_n, \dots, -x_2, -x_1, x_1, x_2, \dots, x_n, 1-x_n, \dots, 1-x_2, 1-x_1, 1+x_1, 1+x_2, \dots, 1+x_n, \dots)$. The inequalities $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1/2$ are equivalent to the condition that this infinite subsequence is increasing.

We note that $\tilde{B}_n$ and $\tilde{C}_n$ are genuinely different groups for $n \geq 3$. The group $\tilde{B}_n$ is the subgroup of $\tilde{C}_n$ where we only reflect over hyperplanes of the form $x_i \pm x_j = k$ and $x_i = k$.

We list the connected positive definite crystallographic Cartan matrices in Table 3. Each of these Cartan matrices has one dimensional kernel, and we label the vertices with a generator of that kernel (up to some issues discussed below) as numbers on the vertices. In each diagram, one vertex is boxed; the parabolic subgroup generated by the unboxed simple generators is the finite group W.

Some comments on notation: The affine group $\tilde{W}$ has one more simple generator than the finite group W. This can be confusing when looking at notation like $\tilde{A}_{n-1}$ and $\tilde{E}_8$ – these refer to groups with n and 9 simple generators respectively. Personally, David Speyer would find it more natural to write W for the affine group and something like W_0 for the finite group, but he has yielded to the standard conventions in these notes. Also, David Speyer would have defined products of affine groups to be affine but, for whatever reason, that isn't standard.

A final technical point: this is a classification of affine Coxeter groups, not a classification of affine Cartan matrices. Different positive semidefinite crystallographic Cartan matrices can be equivalent under $A_{ij} \mapsto \frac{c_i}{c_j} A_{ij}$ and hence give the same affine group. For example, $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right]$ both give rise to $\tilde{A}_1$, which is the infinite dihedral group, and they give isomorphic reflection representations. We note that this changes the kernel vector δ from $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$; we have made one particular choice of δ in each diagram above, specifically, the one where the roots and co-roots are $\left[\begin{smallmatrix} \beta \\ k \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \beta^\vee \\ 0 \end{smallmatrix}\right]$.

The matrices $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right]$ give non-isomorphic root systems and, for those interested in applications to cluster algebras, to quiver representations or to Kac-Moody Lie algebras/groups/quantum groups, it is important to distinguish them. If you need to refer to positive semidefinite crystallographic Cartan matrices, there are standard notations for them which you can find in Kac's Chapter Four of book *Infinite Dimensional Lie Algebras* (Cambridge University Press, 1990). Macdonald's "Affine root systems and Dedekind's η-function" (*Inventiones*, 1972) also classifies positive semidefinite crystallographic Cartan

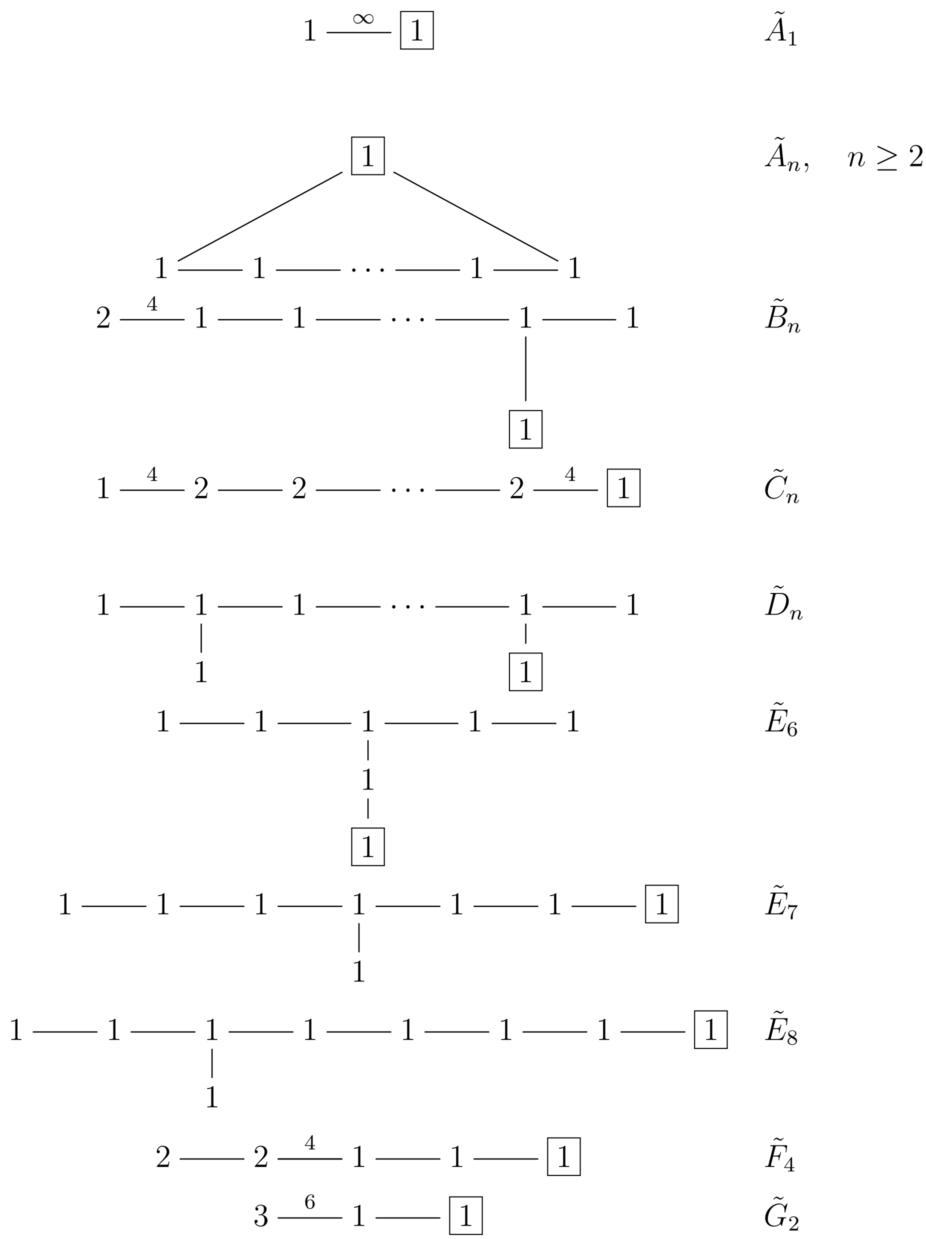


TABLE 3. The affine Coxeter groups

matrices and uses what Professor Speyer considers much better notation, but his notation sadly hasn't caught on.

18. Discrete Euclidean reflection groups are Coxeter groups

Let V_0 be a vector space with a positive definite inner product and V_1 a principal homogenous space for it. Let G be a discrete reflection subgroup of $\mathrm{Euc}(V_1)$. Let D_1 be a connected component of $V_1 \setminus \bigcup \mathrm{Fix}(t)$. Choose an identification of V_1 with V_0 and let the defining inequalities of D_1 be $\{x \in V_0 : \beta_i \cdot x \geq k_i\}$ for some $\beta_i \in V_0$ and some $k_i \in \mathbb{R}$. Let H_i be the hyperplane $\{x \in V_0 : \beta_i \cdot x = k_i\}$. Let s_i be the reflection over H_i and let m_{ij} be the order of $s_i s_j$. Put $\beta_i^\vee = \frac{2}{\beta_i \cdot \beta_i} \beta_i$.

We want to show that G is a Coxeter group with respect to these generators s_i and orders m_{ij}. We note that G is a reflection subgroup of $V_0 \oplus \mathbb{R}$ where s_i corresponds to the root $\left[\begin{smallmatrix} \beta_i \\ -k_i \end{smallmatrix}\right]$ and coroot $\left[\begin{smallmatrix} \beta_i^\vee & 0 \end{smallmatrix}\right]$. These roots and coroots pair by the matrix $\left[\begin{smallmatrix} \beta_i^\vee & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \beta_j \\ -k_j \end{smallmatrix}\right] = \beta_i^\vee \cdot \beta_j$. Setting $A_{ij} = \beta_i^\vee \cdot \beta_j$, we must show that A_{ij} is a Cartan matrix for m_{ij}.

Case I: The order m_{ij} is finite. Then the hyperplanes H_i and H_j are not parallel; let them meet with angle θ_{ij}, measured on the side that contains the polytope D. The angle θ_{ij} must be of the form $\frac{\ell}{m_{ij}}\pi$ with $\mathrm{GCD}(\ell, m_{ij}) = 1$, since $s_i s_j$ has order m_{ij}. If ℓ is not 1, then the subgroup generated by s_i and s_j contains a hyperplane that passes through the wedge between H_i and H_j bounding D, a contradiction. So the angle is $\frac{\pi}{m_{ij}}$. We deduce that A_{ij}, $A_{ji} \leq 0$, and 0 if and only if $m_{ij} = 2$, and that $A_{ij} A_{ji} = 4 \cos^2 \frac{\pi}{m_{ij}}$.

Case II: The order m_{ij} is infinite. Then the hyperplanes H_i and H_j are parallel, and thus the normal vectors β_i and β_j are parallel. Since the half spaces $\{x \in V_0 : \beta_i \cdot x \geq k_i\}$ and $\{x \in V_0 : \beta_j \cdot x \geq k_j\}$ intersect, the vectors β_i and β_j must point in opposite directions, so A_{ij} and A_{ji} are negative. Since β_i, β_j, $\beta_i^\vee$ and $\beta_j^\vee$ all lie on the same line, we must have $A_{ij} A_{ji} = 4$, as desired.

Since the $\left[\begin{smallmatrix} \beta_i \\ -k_i \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \beta_i^\vee & 0 \end{smallmatrix}\right]$ pair by a Cartan matrix for m_{ij}, we know that the group they generate is the abstract Coxeter group $\langle s_i \mid (s_i s_j)^{m_{ij}} \rangle$.

19. Coxeter groups of affine Cartan matrices, basic structure

We will get a fair bit of milage out of the following lemma:

Lemma. Let A be a positive semi-definite connected $n \times n$ Cartan matrix and let $(c_1, c_2, \ldots, c_n)$ be a nonzero vector in the kernel of A. Then all the c_i have (strictly) the same sign.

Define the matrix $B_{ij} = d_i A_{ij}$, so B is symmetric. We note that $B = DA$ where D is the diagonal matrix with entries d_i, so $\mathrm{Ker}(A) = \mathrm{Ker}(B)$. Note that, since B is positive semi-definite, we have $B\vec{c} = 0$ if and only if $\vec{c}^T B \vec{c} = 0$.

Proof. Let I_+, I_0 and I_- be the sets of indices where c_i is > 0, $= 0$ and < 0 respectively. Put $\vec{c}_+ = \sum_{i \in I_+} c_i e_i$ and $\vec{c}_- = -\sum_{i \in I_-} c_i e_i$, so $\vec{c} = \vec{c}_+ - \vec{c}_-$.

Expanding $\vec{c}^T B \vec{c} = 0$, we get

$$\vec{c}_+^T B \vec{c}_+ + \vec{c}_-^T B \vec{c}_- = 2 \vec{c}_+^T B \vec{c}_-.$$

Since B is positive semidefinite, the left hand side is ≥ 0. But the right hand side is $2 \sum_{i \in I_+,\ j \in I_-} c_i c_j B_{ij}$, and every term of this sum is ≤ 0. Comparing the two, we deduce that $B\vec{c}_+ = B\vec{c}_- = 0$.

What we want to show is that one of I_+ and I_- is all of $\{1, 2, \ldots, n\}$ and the other is $\emptyset$. Without loss of generality, suppose that I_+ is nonempty and suppose for the sake of contradiction that $I_+ \neq \{1, 2, \ldots, n\}$. Since Γ is connected, there is some $i \in I_+$ that borders $j \notin I_+$. But then the j-th component of $B\vec{c}_+$ is < 0, a contradiction. □

Corollary. Let A be a positive semi-definite connected $n \times n$ Cartan matrix. Then the signature of B is either $+ + + \cdots +$ or $+ + + \cdots 0$.

If the signature is $++ + \cdots +$ then A is positive definite and corresponds to a finite group, which we already understand. We therefore assume A has signature $+++\cdots 0$. Let $(c_1, \ldots, c_n)$ be in the kernel of A with all $c_i > 0$.

Let $\tilde{W}$ be the Coxeter group corresponding to A. Consider a linear reflection representation V of $\tilde{W}$ corresponding to A with the α_i a basis (hence $D^\circ \neq \emptyset$). Define a symmetric bilinear form on V by $\alpha_i \cdot \alpha_j = d_i A_{ij}$, so this form is positive semidefinite with signature $+++\cdots 0$. The reflection representation preserves this form.

Put $\delta = \sum c_i \alpha_i$. Since the α_i are independent, this is not 0. We have $\langle \alpha_i^\vee, \delta \rangle = 0$ for all i, so all reflections s_i fix δ. So $\tilde{W}$ preserves a symmetric positive semidefinite form with signature $+++\cdots 0$ and a null vector δ, and we can think of it as a subgroup of a Euclidean symmetry group, $\mathrm{Euc}(V_1)$ where $V_1 = \{x \in V^\vee : \langle x, \delta \rangle = 1\}$.

We claim that this subgroup is discrete and, in fact, describe its fundamental domain. We know that a fundamental domain for $\tilde{W}$ acting on $V^\vee$ is $D = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0\}$. So a fundamental domain for the action on V_1 is

$$D \cap V_1 = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0,\ \langle x, \delta \rangle = 1\}.$$

Since the α_i are a basis of V, the inner products with them form coordinates on $V^\vee$, and this is identified with the simplex $\{(y_1, \ldots, y_n) : y_i \geq 0,\ \sum c_i y_i = 1\}$. We are using that the c_i are positive to see that this is a simplex.

Finally, we check that the images of D_1 fill V_1. If not, then there must be some point on the boundary of $\bigcup_{w \in \tilde{W}} wD_1$ where infintely many domains wD_1 accumulate. But all the wD_1 are congruent simplices, so we can't pack infinitely many of them near a point. We conclude that $\bigcup_{w \in \tilde{W}} wD_1 = V_1$ and that the Tits cone is the half space $\{0\} \cup \{x : \langle x, \delta \rangle \geq 0\}$.

20. Affine Coxeter groups as semidirect products

We now know that discrete Euclidean reflection groups are all Coxeter groups, and that the connected ones are of two kinds:

- Finite groups
- Affine groups, meaning groups coming from a connected positive semi-definite Cartan matrix. These have signature $++ + \cdots + 0$.

What remains of our claims from section 17 is to show that the affine groups are precisely the groups of the form $W \ltimes \mathbb{Z}\langle \alpha_i^\vee \rangle$ where W is a finite connected crystallographic Coxeter group and $(\Phi, \Phi^\vee)$ is a corresponding crystallographic root system.

Let W be a finite connected crystallographic Coxeter group and let $(\Phi, \Phi^\vee)$ be a corresponding crystallographic root system in some inner product space V. Set $\tilde{W} := W \ltimes \mathbb{Z}\langle \alpha_i^\vee \rangle$ is clearly a subgroup of $\mathrm{Euc}(V)$. We will show that $\tilde{W}$ is a discrete reflection group.

First, we check that it is a reflection group. Let t be any reflection of W and let $(\beta, \beta^\vee)$ be a corresponding root and coroot. Then the element $(t, \beta^\vee)$ in $W \ltimes \mathbb{Z}\langle \alpha_i^\vee \rangle$ is the orthogonal reflection over $\{x : \langle x, \beta \rangle = 1\}$, and $(t, 0)$ is the orthogonal reflection over $\{x : \langle x, \beta \rangle = 0\}$. Let G be the subgroup of $\tilde{W}$ generated by these reflections; we will show that G is the whole group.

Since G contains all of the $(s_i, 0)$, it contains the subgroup $W \ltimes \{0\} = W$ of $\tilde{W}$. We also compute that $(s_i, \alpha_i^\vee)(t, 0) = (e, \alpha_i^\vee)$, so G contains the translation by $\alpha_i^\vee$. Thus, G contains $\mathbb{Z}\langle \alpha_i^\vee \rangle$. The subgroups W and $\mathbb{Z}\langle \alpha_i^\vee \rangle$ generate $\tilde{W}$.

We will want to know the complete list of reflections in $\tilde{W}$. An element (σ, γ) of $\mathrm{Euc}(V)$ squares to the identity if and only if $\sigma^2 = e$ and $\sigma(\gamma) = -\gamma$. It is a reflection if and only if, additionally, σ is a reflection. So the reflections of $\tilde{W}$ are the elements of the form (t, γ) where t is a reflection of W and $t(\gamma) = -\gamma$. The latter equation shows that we must have $\gamma \in \mathbb{R}\beta_t^\vee$. A type by type check shows that $\mathbb{R}\beta_t^\vee \cap \mathbb{Z}\langle \alpha_i^\vee \rangle = \mathbb{Z}\beta_t^\vee$. (I don't know a conceptual proof of this.) We conclude that the reflections of $\tilde{W}$ are precisely the elements $(t, k\beta_t^\vee)$ for $k \in \mathbb{Z}$. The corresponding fixed hyperplanes are $\{x : \langle x, \beta_t \rangle = k\}$.

In particular, the hyperplanes fall into finitely many parallel classes (one for each $t \in T$), so the hyperplane arrangement is discrete and $\tilde{W}$ is a discrete reflection group. By Section 18, $\tilde{W}$ is a Coxeter group.

Now suppose that $\tilde{W}$ is an affine group: As discussed in Section 19, we can build an inner product space V_0 and principal homogenous space V_1 so that $\tilde{W}$ can be thought of as a subgroup of $\mathrm{Euc}(V_1)$. The reflecting hyperplanes of $\tilde{W}$ are of the form $\{x : \langle x, \beta \rangle = k\}$ for various $\beta \in V_0$ and $k \in \mathbb{R}$. We start with the following lemma:

Lemma. Up to rescaling, only finitely many normal vectors β occur for reflecting hyperplanes in $\tilde{W}$.

Proof. The unit sphere in V_0 is compact, so it is enough to prove a lower bound for the angle between β_1 and β_2 for β_1 and β_2 not parallel. Equivalently, we want to show a lower bound for the angle between non-parallel hyperplanes H_1 and H_2. Since H_1 and H_2 are assumed non-parallel, they meet at some point x in V_1. By the end of Section 19, x lies in wD_1 for some $w \in \tilde{W}$. But w acts by Euclidean symmetries, so we may replace H_1 and H_2 by $w^{-1}H_1$ and $w^{-1}H_2$ to assume that the hyperplanes meet D_1. But $\tilde{W}$ is discrete, so only finitely many hyperplanes meet D_1. □

Now, consider the short exact sequence $1 \to V_0 \to \mathrm{Euc}(V_1) \to O(V_0) \to 1$. Define W to be the image of $\tilde{W}$ in $O(V_0)$ and define $P = \tilde{W} \cap V_0$. So we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P & \longrightarrow & \tilde{W} & \longrightarrow & W & \longrightarrow & 1 \\ & & \cap| & & \cap| & & \cap| & & \\ 1 & \longrightarrow & V_0 & \longrightarrow & \mathrm{Euc}(V_1) & \longrightarrow & O(V_0) & \longrightarrow & 1 \end{array} \qquad (*)$$

Now, W is generated by the images of the reflections of $\tilde{W}$, and our lemma above shows that these reflections only have finitely many images in $O(V_0)$. So W is a finite reflection group, and is thus a finite Coxeter group.

Lemma. Let t be a reflection of W. Then we can lift $\tilde{t}$ to a reflection in $\tilde{W}$.

Note a generic lift of t to $\mathrm{Euc}(V_1)$ will be a glide reflection, so there is something to prove here.

Proof. Let $\tilde{T}$ and T be the sets of reflections in $\tilde{W}$ and W. The quotient map induces a map $\tilde{T} \to T$, which we want to prove is surjective. Let X denote the image of this map.

Since $\tilde{T}$ is conjugacy invariant in $\tilde{W}$, and $\tilde{W} \to W$ is surjective, we see that X is a conjugacy invariant subset of T. Also, $\tilde{T}$ generates $\tilde{W}$ (since $\tilde{W}$ is a reflection group) and $\tilde{W} \to W$ is surjective, so X generates W. The claim now follows if we can show that, if X is a proper conjugacy invariant subset of T then X does not generate W.

I can give a conceptual proof of this statement, which is true for infinite W as well, but it is much faster to just check cases: On the homework, we gave a criterion for when two reflections are conjugate. In types $ADEH$ and $I_2(2k+1)$, all reflections are conjugate, so the claim is immediate. In types B_n, F_4 and $I_2(2k)$, there are two conjugacy classes of reflections. If we take just one conjugacy class of reflection in these types, we generate the following proper subgroups: In B_n, either A_1^n or D_n; in F_4, a subgroup isomorphic to D_4; in $I_2(2k)$, the subgroup $I_2(k)$. $\square$

Lemma. The short exact sequence $1 \to P \to \tilde{W} \to W \to 1$ is semidirect. In other words, $\tilde{W} = W \ltimes P$.

We note that this does not follow from the diagram $(*)$ alone. As a simple example, let τ be the glide reflection $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x+1 \\ -y \end{smallmatrix}\right]$ and consider the group $\langle \tau \rangle$ that τ generates in $\mathrm{Euc}(\mathbb{R}^2)$. Then $\langle \tau \rangle \cap \mathbb{R}^2 = \left[\begin{smallmatrix} 2\mathbb{Z} \\ 0 \end{smallmatrix}\right]$ and the image of $\langle \tau \rangle$ in $O(\mathbb{R}^2)$ is $\left[\begin{smallmatrix} 1 & \\ & \pm 1 \end{smallmatrix}\right]$. So we have a short exact sequence $1 \to \left[\begin{smallmatrix} 2\mathbb{Z} \\ 0 \end{smallmatrix}\right] \to \langle \tau \rangle \to \left[\begin{smallmatrix} 1 & \\ & \pm 1 \end{smallmatrix}\right] \to 1$ where the left group is a discrete translation subgroup of $\mathbb{R}^2$ and the right group is a reflection group of $\mathbb{R}^2$, but the sequence is not semidirect. The reader might think this is because $\left[\begin{smallmatrix} 2\mathbb{Z} \\ 0 \end{smallmatrix}\right]$ does not have full rank in $\mathbb{R}^2$ but this is wrong; let $\tau_1 \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+1 \\ -y \end{smallmatrix}\right]$ and $\tau_2 \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} -x \\ y+1 \end{smallmatrix}\right]$. Then the group generated by these glide reflections sits in a short exact sequence $1 \to \left[\begin{smallmatrix} 2\mathbb{Z} \\ 2\mathbb{Z} \end{smallmatrix}\right] \to \langle \tau_1, \tau_2 \rangle \to \left[\begin{smallmatrix} \pm 1 & \\ & \pm 1 \end{smallmatrix}\right] \to 1$ which is, once again, not semidirect. What we need to show is that we cannot have this happen if our group is generated by reflections, rather than glide reflections.

Proof. Take simple generators s_i of W. Lift them to reflections $\tilde{s}_i$ in $\tilde{W}$. We claim this extends to a right splitting $\tilde{W} \leftarrow W$. We must simply check that the $\tilde{s}_i$ obey the Coxeter relations. Since the $\tilde{s}_i$ are reflections, they obey $\tilde{s}_i^2 = e$. For $i \neq j$, the fixed affine hyperplanes of $\tilde{s}_i$ and $\tilde{s}_j$ meet at the same angle as the fixed central hyperplanes of s_i and s_j, so we have $(\tilde{s}_i \tilde{s}_j)^{m_{ij}} = e$. $\square$

We note that this argument also shows that W can be taken to be a parabolic subgroup of $\tilde{W}$. The fixed planes of the $\tilde{s}_i$ intersect at some point x of V_1. (They are affine hyperplanes with linearly independent normal vectors.) The stabilizer of x in $\tilde{W}$ is the image W for the splitting we just constructed. But the stabilizer of any point in the Tits cone is a parabolic subgroup.

What remains is to show that W is crystallographic and that P is $\mathbb{Z}\langle \alpha_i^\vee \rangle$ for some Crystallographic root system $\Phi^\vee$ of W.

We first note that P is a lattice. P is constructed as an additive subgroup of the vector space V_0. Using that $\tilde{W}$ is discrete, it is easy to see that P is discrete. We must also show that P spans V_1. If not, then P is contained in some proper linear subspace L. But then the finitely many domains wD for $w \in W$ would only reach a bounded distance from L (recall that D is a simplex, and in particular bounded), so the translates of D by $W \ltimes P$ would likely only reach a finite distance from L, contradicting that the $W \ltimes P$ translates of D fill V_1. So P is a lattice.

Since W preserves a lattice, it is crystallographic. Let $\alpha_i^\vee$ be the minimal (-1)-eigenvector of s_i in the lattice P and let $\alpha_i = \frac{2\alpha_i}{\alpha_i \cdot \alpha_i}$. Then we showed earlier that these form crystallographic roots and co-roots for W and that any W-invariant lattice lies between $\mathbb{Z}\langle \alpha_i^\vee \rangle$ and $\mathbb{Z}\langle \alpha_i \rangle^\vee$. But, as computed at the end of Section 16, any reflection in $W \ltimes P$ must lie in the smaller group $W \ltimes \mathbb{Z}\langle \alpha_i^\vee \rangle$. So the claim that $\tilde{W}$ is a reflection group means that we must have $P = \mathbb{Z}\langle \alpha_i^\vee \rangle$ as claimed. $\square$

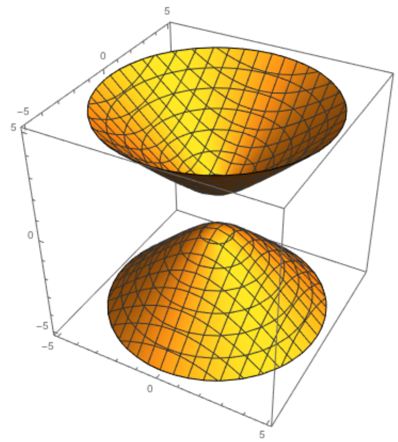


FIGURE 14. The hyperboloid $Q = Q_+ \cup Q_-$

21. Hyperbolic Coxeter groups

Here is a summary of the different types of Coxeter groups that give pretty pictures:

- Positive definite bilinear forms correspond to finite Coxeter groups which look like triangulations of the sphere S^{n-1}.
- Positive semi-definite bilinear forms correspond to affine Coxeter groups which look like triangulations of the affine space $\mathbb{R}^{n-1}$.
- Bilinear forms with signature $(++\ldots+-)$ correspond to hyperbolic Coxeter groups which look like triangulations of the hyperbolic space $\mathbb{H}^{n-1}$.

We've seen two of these, and we now sketch the third.

Hyperbolic geometry: Let V be an n-dimensional real vector space with a symmetric bilinear form, the dot product $\cdot$ with signature $(+^{n-1}-)$. Let $Q = \{\vec{v} \in V : \vec{v} \cdot \vec{v} = -1\}$. This is a hyperboloid of two sheets. Call this hyperboloid Q and denote its sheets by Q_+ and Q_- respectively. Each sheet, as a manifold, is an open ball of dimension $n-1$.

For $\vec{v} \in Q_+$, we have the tangent space $T_{\vec{v}}Q_+ = \{\vec{x} \in V : \vec{v} \cdot \vec{x} = 0\} = \vec{v}^\perp$. Restricting $\cdot$ to $T_{\vec{v}}Q$ gives a positive definite dot product on $T_{\vec{v}}Q$ (because $V = \mathbb{R}\vec{v} \oplus (\vec{v})^\perp$). So Q_+ is naturally a Riemannian manifold, commonly called the hyperbolic plane/space/disk.

Given $\vec{v} \in V$ with $\vec{v} \cdot \vec{v} > 0$, we can reflect over $\vec{v}^\perp$ in the following way:

$$\vec{x} \mapsto \vec{x} - 2\frac{\vec{v} \cdot \vec{x}}{\vec{v} \cdot \vec{v}}\vec{v}.$$

This takes Q_+ to itself. Thus, we can talk about hyperbolic reflection groups, which are order two isometries fixing hyperplanes in hyperbolic space.

If $n = 3$ then Q_+ is a two dimensional disc. There are two standard ways of explicitly identifying Q_+ with the unit disc in $\mathbb{R}^2$.

The Klein model Take an affine plane, K, separating the origin from Q_+. Plot $\vec{x} \in Q_+$ at the intersection of $\mathbb{R}\vec{x}$ with K. See Figure 15. The picture will be contained within an open ball and hyperplanes are represented as affine hyperplanes in the open ball.

Poincare model Choose $\vec{v} \in Q_+$. Plot $\vec{x} \in Q_+$ at the intersection of the line through $-\vec{v}$ and $\vec{x}$ with $T_{\vec{v}}Q_+$. See Figure 16. In the Poincare model, lines are arcs of circles perpendicular to the boundary of the disk. The Poincare model is conformal, meaning that

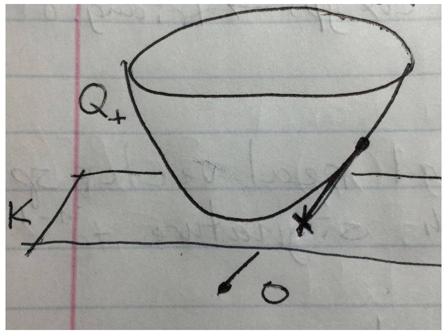


FIGURE 15. The point x represents the image of a point in the Klein model

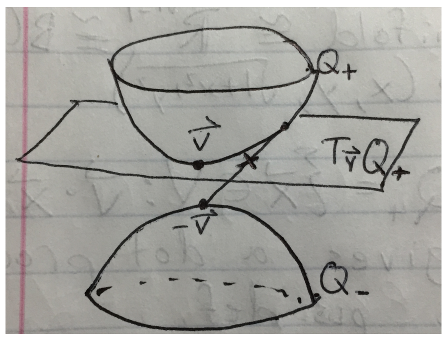


FIGURE 16. The point x represents the image of a point in the Poincare model

angles are correct. It also puts things nearer to the center of the disk, so the diagram can render more detail.

Example. Consider the Cartan matrix and reflection group given as follows:

$$A = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \quad W = \langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = 1 \rangle$$

The left and right sides of Figure 17 draw this arrangement in the Klein and Poincare models.

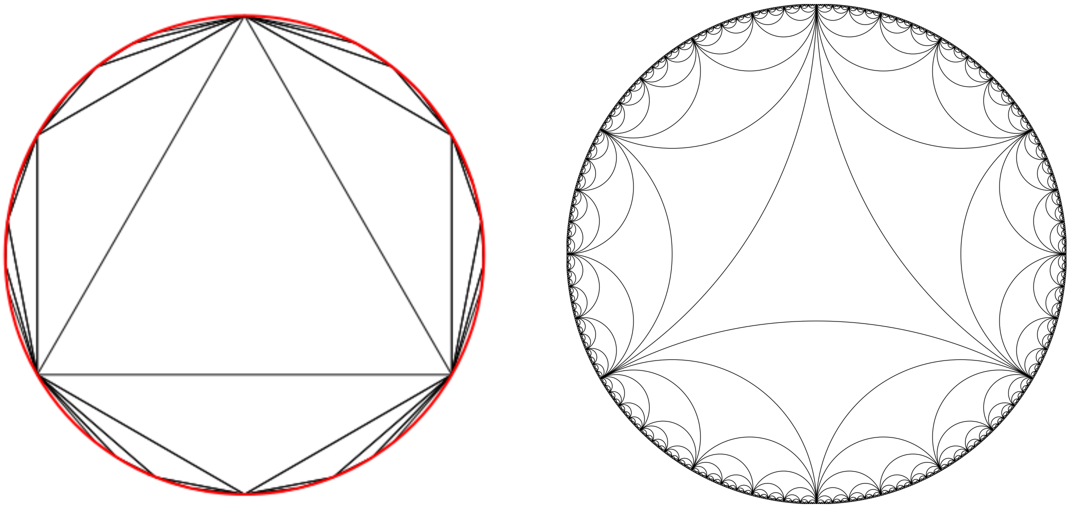


FIGURE 17. A Triangulation in the Klein model and the Poincare models

If we have $\alpha_1, ..., \alpha_n \in V$ with $\alpha_i \cdot \alpha_i = 2$ and $\alpha_i \cdot \alpha_j \in \{-2\cos\frac{\pi}{m} : m \geq 2\} \cup (-\infty, -2]$ and $\{x : \alpha_i \cdot x > 0\} \neq \emptyset$ we'll get a Coxeter group acting on Q_+. We can do this if A is symmetric with signature $(++...+-)$. Take V to be the vector space on basis $\{\alpha_i\}$ and define $\cdot$ by $\alpha_i \cdot \alpha_j = A_{ij}$.

Example. If we want $m_{12} = m_{13} = m_{23} = 4$ (i.e. triangles with all angles equal to $\frac{\pi}{4}$) then use the Cartan matrix

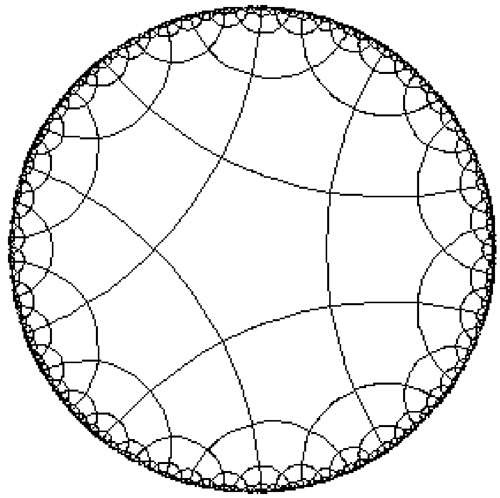
$$\begin{bmatrix} 2 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 2 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 \end{bmatrix}$$

We can also get hyperbolic tilngs by polytopes other than simplices. Suppose that A is a symmetric Cartan matrix with signature $+^{n-1}0^r-$. Then we can find $n+r$ vector in V which pair by A, and thus get a polytope in hyperbolic $n-1$ space with $n+r$ facets.

Example. Suppose we want to tile the disk with pentagons whose angles are all $\frac{\pi}{2}$. Then we need $m_{12} = m_{23} = ... = m_{51} = 2$ and $m_{13} = m_{24} = ... = \infty$. So our Cartan matrix should be of the form

$$A = \begin{bmatrix} 2 & 0 & -\alpha & -\alpha & 0 \\ 0 & 2 & 0 & -\alpha & -\alpha \\ -\alpha & 0 & 2 & 0 & -\alpha \\ -\alpha & -\alpha & 0 & 2 & 0 \\ 0 & -\alpha & -\alpha & 0 & 2 \end{bmatrix} \text{ with } \alpha \geq 2$$

We can achieve signature $++00-$ if we take $\alpha = 1+\sqrt{5}$. (More generally, we could choose 5 distinct values for off diagonal terms, constrained to make this matrix have rank 3, and get hyperbolic pentagons with angles $\frac{\pi}{2}$ and sides of different lengths.) Here is the resulting tiling.



However, there is a technical point to worry about. How will D relate to Q_+? In all our examples so far, $\cdot$ is ≤ 0 on D and $\mathrm{Tits}(W)$ filled up the cone $\mathbb{R}_{\geq 0}Q_+$.

Here is the answer: We always have $\mathbb{R}_{\geq 0}Q_+ \subseteq \mathrm{Tits}(W)$. The domain $D \setminus \{0\}$ is in the interior of $\mathbb{R}_{\geq 0}Q_+$ if and only if the $(n-1)\times(n-1)$ principal minors are positive definite or, equivalently, if the parabolic subgroups are finite. The domain D is in the closure of $\mathbb{R}_{\geq 0}Q_+$ if and only if the $(n-1)\times(n-1)$ principal minors are positive semidefinite or, equivalently, if the parabolic subgroups are finite or affine.

If some of these principal minors have $-$ part to their signature, then D sticks out of the outside of Q_+.

Example. Let $A = \begin{bmatrix} 2 & -2.2 & -2.2 \\ -2.2 & 2 & -2.2 \\ -2.2 & -2.2 & 2 \end{bmatrix}$. The corresponding Coxeter has $m_{12} = m_{13} = m_{23} = \infty$, just as in Figure 17. Figure 18 depicts this example in the Klein model. There is no natural way to draw this picture in the Poincare model, because the Poincare model has no natural place to draw points not coming from Q_+.

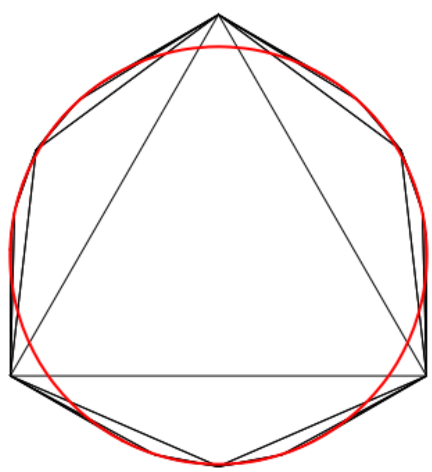


FIGURE 18. Klein model for an example with negative signature

Notational warning: A Coxeter group is only called ***hyperbolic*** if D does not stick out of $\mathbb{R}_{\geq 0}Q_+$. The hyperbolic Coxeter groups have been classified, but the list is quite long.