Problem Set 1: Due Friday, September 13

You should be able to begin work on all of these problems now! See the course website for homework policy.

1. Read the first five sections of the Course Notes ("Introduction" through "The geometry of two reflections"). Suggest either something to be improved, or a question these notes raise.

2. The reverse of this problem set depicts the A_3 hyperplane arrangement in stereographic projection. Here D is the region $x_1 < x_2 < x_3 < x_4$, and s_1, s_2 and s_3 are the permutations $(1\ 2)$, $(2\ 3)$ and $(3\ 4)$.
   (a) Label the region on the diagram where the point $(2019, 3, 665, 17)$ is located.
   (b) Give the defining inequalities of this region.
   (c) Give a word w in the generators s_1, s_2, s_3 such that this point lies in wD.

3. (a) The B_n hyperplane arrangement consists of the following list of hyperplanes in $\mathbb{R}^n$: $x_i \pm x_j = 0$ for $1 \leq i < j \leq n$ and $x_i = 0$ for $1 \leq i \leq n$. How many regions does the complement of these hyperplanes have?
   (b) The D_n hyperplane arrangement is the subset of the B_n arrangement consisting of the hyperplanes $x_i \pm x_j = 0$ for $1 \leq i < j \leq n$. How many regions does the complement of these hyperplanes have?

4. Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$ and f_1, f_2, f_3 be the standard basis of $\mathbb{R}^3$. The A_3 root system consists of the 12 vectors of the form $e_i - e_j$ with $i \neq j$; the D_3 root system consists of the 12 vectors of the form $\pm f_i \pm f_j$ with $i \neq j$. Give a linear isomorphism from $\mathbb{R}^3$ to $\mathrm{Span}(e_i - e_j) \subset \mathbb{R}^4$ taking the D_3 root system to the A_3 root system.

5. We recall/preview the following definitions from class: Let Φ be a finite collection of vectors in $\mathbb{R}^n$, such that $\alpha \in \Phi$ implies $-\alpha \in \Phi$. Let $\rho \in \mathbb{R}^n$ such that $\langle \alpha, \rho \rangle \neq 0$ for any $\alpha \in \Phi$. We define the set of ***positive roots***, Φ^+ to be those roots $\alpha \in \Phi$ with $\langle \alpha, \rho \rangle > 0$. We define a positive root to be ***simple*** if it is not a positive linear combination of other positive roots.

   In the following cases, describe the positive roots and the simple roots. We write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{R}^n$.
   (a) Φ is all vectors in $\mathbb{R}^n$ of the forms $\pm e_i \pm e_j$ (with $i \neq j$) and $\pm e_i$. Take $\rho = (1, 2, 3, \ldots, n)$.
   (b) Φ is all vectors in $\mathbb{R}^n$ of the forms $\pm e_i \pm e_j$ (with $i \neq j$). Take $\rho = (1, 2, 3, \ldots, n)$.
   (c) Φ is all vectors in $\mathbb{R}^4$ of the forms $\pm e_i \pm e_j$ (with $i \neq j$), $\pm e_i$ and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. Take $\rho = (1, 2, 4, 8)$.
   (d) Φ is all vectors in $\mathbb{R}^8$ of the forms $\pm e_i \pm e_j$ and $\frac{1}{2}(\pm 1, \pm 1, \ldots, \pm 1)$ with an odd number of -1's. Take $\rho = (1, 2, 4, \ldots, 2^7)$.

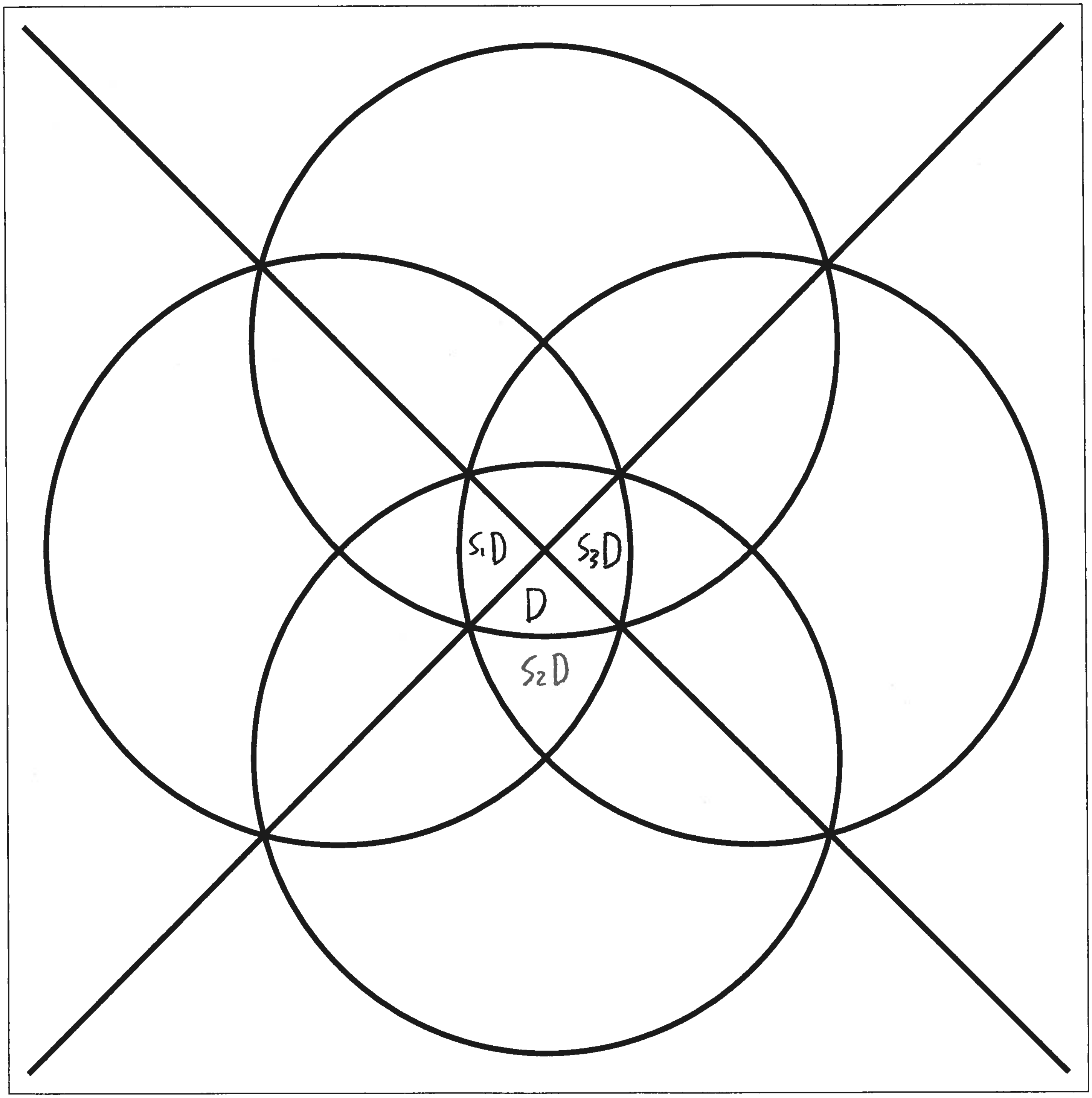
6. Let V be a two dimensional real vector space with basis α_1, α_2 and let $\alpha_1^\vee$ and $\alpha_2^\vee \in V^\vee$ be the vectors such that $\langle \alpha_1^\vee, \alpha_1 \rangle = \langle \alpha_2^\vee, \alpha_2 \rangle = 2$ and $\langle \alpha_1^\vee, \alpha_2 \rangle = \langle \alpha_2^\vee, \alpha_1 \rangle = -2$. Let s_i act on V by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$ (note that $s_1^2 = s_2^2 = 1$). Let W be the group generated by s_1 and s_2.
   (a) Give a simple description of the orbits of α_1 and α_2 under W.
   (b) Let $D = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0\}$. Draw and label D, $s_1 D$, $s_2 D$, $s1s2D$, $s_2 s_1 D$.
   (c) Give a simple description of $\bigcup_{w \in W} wD$.
   (d) Now suppose that $\langle \alpha_1^\vee, \alpha_2 \rangle = \langle \alpha_2^\vee, \alpha_1 \rangle = -3$ instead of -2. Repeat parts (a), (b) and (c).

7. This problem introduces the affine symmetric group $\tilde{A}_{n-1}$, which will be an important example of an infinite Coxeter group.

   Fix an integer $n \geq 3$. Define $\tilde{S}_n$ to be the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ which obey $w(i + n) = w(i) + n$, made into a group under composition. For $1 \leq i \leq n$, define the element $s_i \in \tilde{S}_n$ by

$$s_i(x) = \begin{cases} x + 1 & x \equiv i \bmod n \\ x - 1 & x \equiv i + 1 \bmod n \\ x & \text{otherwise} \end{cases}.$$

   Define $\tilde{A}_{n-1}$ be $\{w \in \tilde{S}_n : \sum_{i=1}^n w(i) = \sum_{i=1}^n i\}$.
   (a) What is the order of $s_i s_j$?
   (b) Show that $\tilde{A}_{n-1}$ is a subgroup of $\tilde{S}_n$ and is generated by the s_i.



S_1D
S_3D
D
S_2D