Problem Set 10: Due Friday, November 22

See the course website for homework policy.

1. Please read the course notes for November 6-20 and suggest a question or something that could be improved.

2. This studies distributive lattices from the point of view of someone who already understands semidistributive lattices. This is out of the historical order, but interesting. Recall that a L lattice is said to be distributive if, for all u, v, w in L, we have

$$u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w) \text{ and } u \wedge (v \vee w).$$

Let L be a finite lattice. In this problem, we will show that the following are equivalent:

(1) L is distributive.

(2) L is semidistributive and, if (x_1, y_1) forces (x_2, y_2) then (x_1, y_1) and (x_2, y_2) are slide equivalent.

(3) L can be obtained from the trivial lattice by repeatedly doubling intervals of the form $[\hat{0}, x]$.

(4) There is a finite poset P such that L is isomorphic to the lattice of lower order ideals of P.

Throughout the problem, let L be a finite lattice.

(a) Assume (1). Let $x \lessdot y$ in L and let $x' = x \vee z$ and $y' = y \vee z$ for some $z \in L$. Suppose that $x' \neq y'$. Show that $x' \lessdot y'$. Hint: Suppose that $x' < w < y'$. Consider $(x' \vee y) \wedge w$.

(b) Show that (1) implies (2).

(c) Assume (2). Show that L is congruence uniform.

(d) Continue to assume (2) and let a cover $\hat{0}$. Show that a is join irreducible. Let L' be the lattice obtained by contracting the slide equivalence class of $(0, a)$. Show that $L \cong L'[\ [\hat{0}', x']\ ]$ for some $x' \in L'$. Conclude (3).

(e) Show that (3) implies (4). Hint: Each doubling will add one new element to P.

(f) Show that (4) implies (1).

Turn over for one more fun problem.

3. This question introduces a fun sort of lattice which I hope to talk about later. Right now, the connection to Coxeter groups should be unclear.

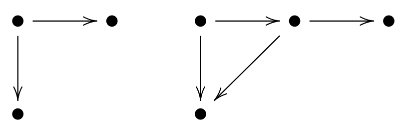
   Let G be a finite acyclic directed graph in which every vertex has even out degree. (Acyclic means that there are no cyclically oriented cycles.) Let V be the vertex of G. For $x \in V$, let $N(x)$ be the set of $y \in V$ such that x has an edge $x \to y$. We define a ***consensus set*** of G to be a subset X of the vertices of G such that, for all vertices $x \in V$:

   - If more than half of the vertices of $N(x)$ are in X, then $x \in X$.
   - If less than half of the vertices of $N(x)$ are in X, then $x \notin X$.

   Note that, if precisely half of the vertices of $N(x)$ are in X, then we impose no condition.

   We write $C(G)$ for the set of consensus sets of G and partially order $C(G)$ by containment.

   (a) Draw the Hasse diagram of $C(G)$ for the acyclic graphs G drawn below:

   (b) Show that, if X is a consensus set, so is $V \setminus X$.

   (c) Let Y be any subset of V. Let Y_1 be $Y \cup \{x : |N(x) \cap Y| > \frac{1}{2}|N(x)|\}$, let $Y_2 = Y_1 \cup \{x : |N(x) \cap Y_1| > \frac{1}{2}|N(x)|\}$, let $Y_3 = Y_2 \cup \{x : |N(x) \cap Y_2| > \frac{1}{2}|N(x)|\}$ etcetera and let $\overline{Y} = \bigcup_j Y_j$. Show that, if X_1 and X_2 are consensus sets, then $\overline{X_1 \cup X_2}$ is a consensus set.

   (d) Show that $C(G)$ is a lattice.

   (e) Let $X \subset Y$ be a cover of $C(G)$. Show that $Y \setminus X$ is a singleton.

   (f) ~~Let $X_1 \lessdot Y_1$ and $X_2 \lessdot Y_2$ be covers of $C(G)$. Show that (X_1, Y_1) and (X_2, Y_2) are slide equivalent if and only if (1) there is an element z of V with $Y_1 = X_1 \cup \{z\}$, $Y_2 = X_2 \cup \{z\}$ and $X_1 \cap N(z) = X_2 \cap N(z)$.~~

   (g) ~~Show that lattices of the form $C(G)$ are congruence uniform.~~

Parts (f) and (g) are removed because at least one of them is false and I am not confident I have found all my errors.