Problem Set 11: Due **Monday**, December 9

See the course website for homework policy.

1. Please read the course notes for November 22-27 and suggest a question or something that could be improved.

2. This is a lemma we'll want on Wednesday or Friday: Let W be a Coxeter group, let s_i and s_j be in W and let $w \in W$ have s_i as a left ascent and s_j as a right ascent. Suppose also that $s_i w \neq w s_j$. Show that s_i is a left ascent of $w s_j$ and s_j is a right ascent of $s_i w$.

3. As usual, let W be a Coxeter group and V a reflection representation. Assume that both the α_i and $\alpha_i^\vee$ are linearly independent. Let $x \in V$ and let Wx be the W-orbit of x. Let G be the directed graph with vertex set Wx and edge $ty \to y$ for $y \in Wx$ whenever $y - ty$ is a positive multiple of β_t.

   (a) Sketch Wx for the cases $(W, x) = (A_3, e_1 + e_2)$ and $(W, x) = (B_3, e_1 + e_2)$.

   (b) Show that G is acyclic. (Hint: Use a vector in D°.)

   (c) Let P be the poset obtained by transitive closure of the directed acyclic graph G. Draw the Hasse diagram of P for the cases $(W, x) = (A_3, e_1 + e_2)$ and $(W, x) = (B_3, e_1 + e_2)$.

4. Let k be a field. For an invertible $n \times n$ matrix g with entries in k and $0 \leq i, j \leq n$, let $r_{ij}(g)$ be the rank of the $i \times j$ submatrix in the lower left of g. We consider a $0 \times j$ or $i \times 0$ matrix to have rank 0.

   (a) Prove the following relations:

      1. We have $r_{0j}(g) = r_{j0}(g) = 0$ and $r_{nj}(g) = r_{jn}(g) = j$.
      2. We have $0 \leq r_{(i+1)j}(g) - r_{ij}(g),\ r_{i(j+1)}(g) - r_{ij}(g) \leq 1$.
      3. If $r_{(i+1)j}(g) = r_{i(j+1)}(g) = r_{ij}(g) + 1$, then $r_{(i+1)(j+1)}(g) = r_{ij}(g) + 2$.

   (b) Let r_{ij}, for $0 \leq i, j \leq n$, be an array of nonnegative integers obeying the conditions of part (a). Show that there is a unique permutation matrix w such that $r_{ij} = r_i(w)$.

   (c) We write B for the group of invertible upper triangular $n \times n$ matrices. Let $g \in \mathrm{GL}_n(k)$ and let w be the unique permutation matrix with $r_{ij}(w) = r_{ij}(g)$. Show that there are matrices b_1 and $b_2 \in B$ such that $g = b_1 w b_2$.