Problem Set 2: Due Friday, September 20

See the course website for homework policy.

1. Read the Course Notes from "Coxeter Groups, Cartan matrices, roots and hyperplanes" through "Consequences of the Key Lemma"). Suggest either something to be improved, or a question these notes raise.

2. Let V be a finite dimensional real vector space equipped with a positive definite symmetric bilinear form $\cdot$. Let Λ be a discrete additive subgroup of V with $\mathrm{Span}_{\mathbb{R}}(\Lambda) = V$. Define G to be the group of linear transformations $g : V \to V$ with $g(u) \cdot g(v) = u \cdot v$ and $g(\Lambda) = \Lambda$.
   (a) Show that G is finite.
   (b) Show that, for $g \in G$, we have $\operatorname{Tr} g \in \mathbb{Z}$.
   (c) Let $\dim V = 2$ and let $g \in G$. Show that g has order 1, 2, 3, 4 or 6.

3. Let W be the subgroup of $\mathrm{GL}_3(\mathbb{R})$ generated by the reflections

$$s_1 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad s_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Let $D = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0\}$.
   (a) Compute the rays of the (polyhedral) cones D, $s_1 D$, $s_1 s_2 D$.
   (b) Show that W preserves the (round) cone $C = \{(x_1 : x_2 : x_3) : x_1 x_2 + x_1 x_3 + x_2 x_3 \geq 0\}$.
   (c) Sketch how D, $s_1 D$ and $s_1 s_2 D$ lie in C.

4. For $1 \leq i < j \leq n$, let $m_{ij} \in \{2, 3, \ldots, \infty\}$. Let W be the group generated by $s_1, s_2, \ldots, s_n$ subject to relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$. Define a graph Γ_2 with vertex set 1, 2, …, n and an edge (i, j) if m_{ij} is odd, where ∞ is considered even.
   (a) Show that the number of homomorphisms $W \to \{\pm 1\}$ is $2^{\#(\text{connected components of } \Gamma_2)}$.
   (b) Show that s_i and s_j are conjugate in W if and only if i and j are in the same connected component of Γ_2

5. Let V be a vector space with basis $\alpha_1, \ldots, \alpha_n$. Let A be an $n \times n$ matrix with $A_{ii} = 2$ and $A_{ij} = 0$ if and only if $A_{ji} = 0$. Let $\alpha_i^\vee$ in $V^\vee$ be the vector with $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$. Define $s_i : V \to V$ by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. Let W be the group generated by the s_i.
   (a) Suppose that there is symmetric bilinear form $(\ , \ )$ on V which is invariant under W, and where $(\alpha_i, \alpha_i) \neq 0$. Defining $d_i = (\alpha_i, \alpha_i)$, show that $d_i A_{ij} = d_j A_{ji}$.
   (b) Conversely, suppose that there exist positive real numbers d_i such that $d_i A_{ij} = d_j A_{ji}$. Show that we can define a symmetric bilinear form $(\ , \ )$ on V with $(\alpha_i, \alpha_i) = d_i$ such that s_i is the orthogonal reflection in α_i.
   (c) Let Γ be the graph with vertices 1, 2, …, n and an edge (i, j) for $i \neq j$ if $A_{ij} \neq 0$. If Γ is a tree, show that there always exist $d_i > 0$ with $d_i A_{ij} = d_j A_{ji}$.

6. Let $n \geq 3$ be a positive integer. Let p be a real number larger than 1. Let V be the vector space of sequences $(a_i)_{i \in \mathbb{Z}}$ such that $a_{i+n} = p a_i$. Let the group $\tilde{A}_{n-1}$ (see previous problem set) act on V by $w(a)_i = a_{w^{-1}(i)}$. The funny inverse is to make it a left action.
   (a) Choose a basis for V, and write the matrices of $s_1, s_2, \ldots, s_n$ in your basis.
   (b) Give explicit vectors $\alpha_i \in V$ and $\alpha_i^\vee \in V^\vee$ such that $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. Choose your signs such that $\langle \alpha_i^\vee, \ \rangle$ is positive on the sequence $(p^{i/n})$.
   (c) Compute the Cartan matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.
   Let $D = \{x \in V : \langle \alpha_i^\vee, x \rangle \geq 0, \ 1 \leq i \leq n\}$. Let $T = \{(x_i) \in V : x_i > 0\}$.
   (d) For $n = 3$, sketch the intersections of T, D and the hyperplanes $\alpha_i^\perp$, with the unit sphere.
   (e) Prove that $\bigcup_{w \in \tilde{A}_{n-1}} wD = T \cup \{0\}$.