Problem Set 3: Due Friday, September 27

See the course website for homework policy.

1. Read the Course Notes from "Reflections transposition and roots" through "Parabolic subgroups". (As of Sept. 18, this means reading to the end of the notes, but I hope to add one or two more sections.) Suggest either something to be improved, or a question these notes raise.

2. In this problem, we fulfill a promise made in class: Make our usual definitions, and assume that D° is nonempty. Let $t \in W$ act on $V^\vee$ by a reflection. We will show that t is of the form $w s_i w^{-1}$ for some $w \in W$ and some Let $t = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and let $H = \mathrm{Fix}(t)$. Put $v_k = s_{i_1} s_{i_2} \cdots s_{i_k}$.
   (a) Show that H does not pass through any of the $v_k D^\circ$.
   (b) Show that H is the wall along which $v_{k-1} D$ borders $v_k D$ for some k.
   (c) Deduce that $t = v_{k-1} s_{i_k} v_{k-1}^{-1}$.

3. Let W be a finite Coxeter group. The longest element, w_0, is the unique element of W satisfying $w_0 D = -D$. Describe the longest element w_0, and its action on V, when W is of types A_n, B_n and D_n.

4. Let W be a Coxeter group with s_i and m_{ij} as usual. Given a word in W, the (i,j) ***braid move*** is to replace the substring $\overbrace{s_i s_j s_i s_j \cdots}^{\text{length } m_{ij}}$ by $\overbrace{s_j s_i s_j s_i \cdots}^{\text{length } m_{ij}}$. Let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ be two reduced words with the same product w. The aim of this problem is to show that we can transform $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ to $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ using only braid moves.

   Our proof is by induction on ℓ, so assume we have proven the result for any pair of words of shorter length.
   (a) If $i_1 = j_1$, show we are done.
   Suppose from now on that $i_1 \neq j_1$. We abbreviate $i_1 = i$, $j_1 = j$ and $m_{ij} = m$.
   (b) Show that all m reflections in $\langle s_i, s_j \rangle$ are inversions of w. (Hint: Geometry!)
   (c) Show that there is a reduced word for w of the form $\overbrace{s_i s_j s_i s_j \cdots}^{\text{length } m} s_{k_{m+1}} s_{k_{m+2}} \cdots s_{k_\ell}$.
   (d) Conclude the proof.

5. This problem describes a different representation of $\tilde{A}_{n-1}$ from the one on Problem Set 2.

   Let $n \geq 3$ be a positive integer. Let V be the vector space of sequences $(a_i)_{i \in \mathbb{Z}}$ such that $a_{i+n} - a_i$ is a constant independent of i. Let $\tilde{A}_{n-1}$ act on V by $w(a)_i = a_{w^{-1}(i)}$.
   (a) Choose a basis for V, and write the matrices of s_1, s_2, ..., s_n in your basis.
   (b) Give explicit vectors $\alpha_i \in V$ and $\alpha_i^\vee \in V^\vee$ such that $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. Choose your signs such that $\langle \alpha_i^\vee, \ \rangle$ is positive on the point $x_i = i$.
   (c) Compute the Cartan matrix $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.
   Once again, let $D = \{x \in V^\vee : \langle x, \alpha_i \rangle \geq 0,\ 1 \leq i \leq n\}$.
   Let $\bar{V}$ be the quotient of V by the vector space of constant sequences. Let $\bar{V}_1$ be the affine subspace $a_{i+n} = a_i + 1$ of $\bar{V}$. Note that $\dim \bar{V}_1 = n - 1$, which means we can draw it for $n = 3$. I'll write $\bar{D}$ for the image of D in $\bar{V}$ and $\bar{D}_1$ for the intersection $\bar{D} \cap \bar{V}_1$.
   (d) For $n = 3$, draw $\bar{D}_1$, $s_1 \bar{D}_1$, $s_2 \bar{D}_1$, $s_3 \bar{D}_1$, $s1s2\bar{D}_1$, $s_2 s_1 \bar{D}_1$ and $s_1 s_2 s_1 \bar{D}_1$ inside $\bar{V}_1$. Draw the hyperplanes fixed by s_1, s_2, s_3.
   (e) Show that $\bar{V}_1 = \bigcup_{w \in \tilde{A}_{n-1}} w \bar{D}_1$.