Problem Set 4: Due Friday, October 4

See the course website for homework policy.

1. Read the Course Notes from "Crystallographic Groups" to "Hyperbolic Coxeter Groups". Suggest either something to be improved, or a question these notes raise.

2. This problem describes a model for the group $\tilde{C}_n$ similar to our model for $\tilde{A}_{n-1}$ in terms of affine permutations. We define $\tilde{C}_n$ by its Coxeter diagram:

$$s_0 \overset{4}{\text{———}} s_1 \text{———} \cdots \text{———} s_{n-1} \overset{4}{\text{———}} s_n$$

Let $Z = \{k + 1/2 : k \in \mathbb{Z}\}$. Let $\tilde{S}^C_n$ be the set of bijections $w : Z \to Z$ such that

$$w(z + 2n) = w(z) + 2n \text{ and } w(-z) = -w(z).$$

Our goal will be to show $\tilde{S}^C_n \cong \tilde{C}_n$.

(a) Describe a map $\tilde{C}_n \to \tilde{S}^C_n$ by describing where to send the generators $s_0, s_1, \ldots, s_n$.

Let $V^\vee$ be the vector space of functions $a : Z \to \mathbb{R}$ obeying $a(-z) = -a(z)$ and such that $a(z+2n) - a(z)$ is a constant independent of z.

(b) Choose a basis for $V^\vee$ and write down the action of $s_0, \ldots, s_n$ in that basis. Write $s_i(x)$ in the form $x - \langle \alpha_i, x\rangle \alpha_i^\vee$ for some roots α_i and $\alpha_i^\vee$.

(c) Check that the α_i and $\alpha_i^\vee$ pair by the Cartan matrix for $\tilde{C}_n$.

(d) Show that $\tilde{C}_n \to \tilde{S}^C_n$ is an isomorphism.

3. Let W be a Coxeter group with generators $s_1, s_2, \ldots, s_n$. For $I \subseteq [n]$, we defined W_I to be the subgroup generated by $\{s_i : i \in I\}$. We define ${}^I W$ by

$${}^I W = \{w \in W : s_i \text{ is a left ascent of } w \text{ for all } i \in I\}$$

(a) With our usual definitions, and assuming that D° is nonempty, show that $wD^\circ \subset D_I^\circ$ if and only if $w \in {}^I W$.

We showed in class that, for each $w \in W$, there is an element $w_I \in W_I$ such that $\mathrm{inv}(w) \cap W_I = \mathrm{inv}(w_I)$. Write $w = w_I {}^I w$.

(b) Show that $w_I {}^I w$ is the unique way to factor w as uv with $u \in W_I$ and $v \in {}^I W$.

We similarly define

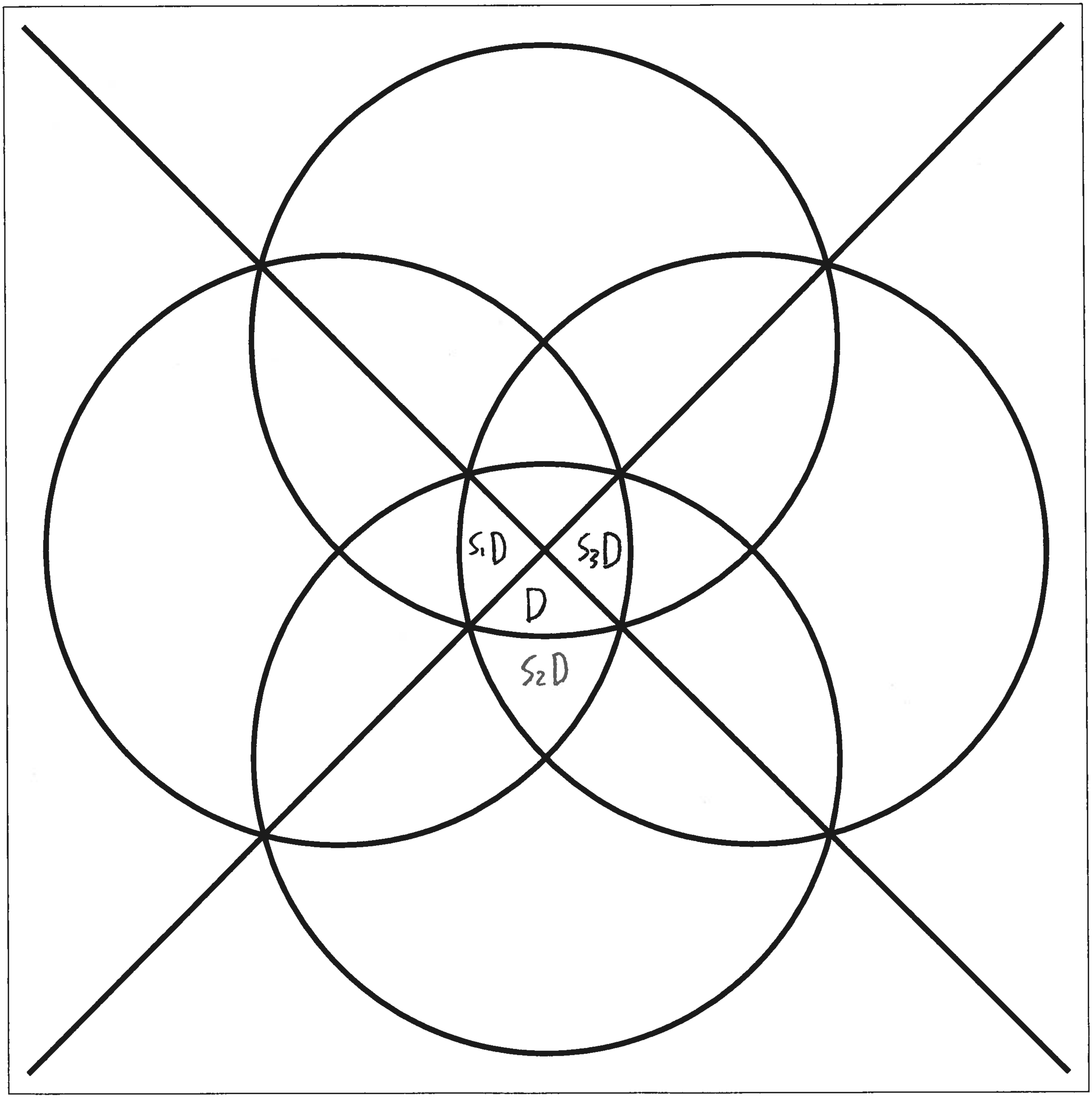
$$W^I = \{w \in W : s_i \text{ is a right ascent of } w \text{ for all } i \in I\}.$$

We put ${}_I w = \left((w^{-1})_I\right)^{-1}$ and $w^I = \left({}^I(w^{-1})\right)^{-1}$, so $w = w^I {}_I w$.

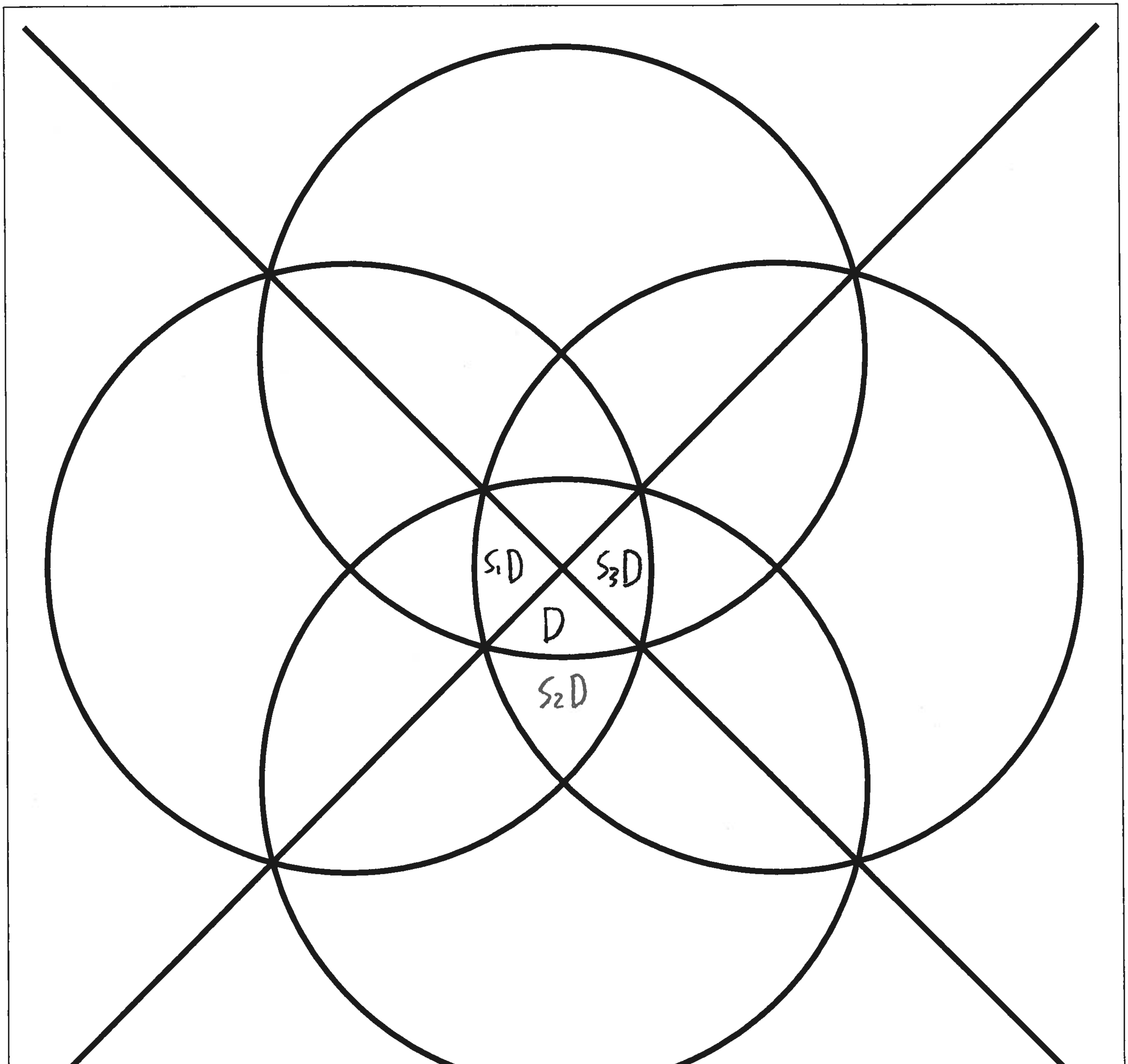
(c) The next two pages of this problem set feature the hyperplane arrangement A_3 (drawn in stereographic projection) with regions D, $s_1 D$, $s_2 D$, $s_3 D$ labeled. Set $I = \{1, 3\}$.

On the first page, use four colors to indicate which w's have w_I equal to 1, s_1, s_3 or $s_1 s3$.

On the second page, outline the cosets uW_I and color according to the value of ${}_I w$.



S1D
S3D
D
S2D



S_1D
S_3D
D
S_2D