Problem Set 5: Due Friday, October 11

See the course website for homework policy.

1. Let C_n be the group of signed $n \times n$ permutation matrices for $n \geq 3$. I number the generators of C_n as s_1, s_2, ..., s_n so that $m_{12} = 4$ and $m_{23} = m_{34} = \cdots = m_{(n-1)n} = 3$. Let G be $C_n \ltimes \mathbb{Z}^n$ with C_n acting on $\mathbb{Z}^n$ by these matrices; we consider G as a group of Euclidean symmetries of $\mathbb{R}^n$. In this problem, we will play with subgroups of G to get experience with Euclidean symmetry groups.
   (a) Show that G is generated by affine reflections. Is it $\tilde{C}_n$ or $\tilde{B}_n$?
   (b) Let G_1 be the subgroup $D_n \ltimes \mathbb{Z}^n$ of $C_n \ltimes \mathbb{Z}^n$. Show that G_1 is not generated by affine reflections.
   (c) Let G_2 be the subgroup of G_1 generated by affine reflections. Show that G_2 is $D_n \ltimes \Lambda$ for a sublattice Λ of $\mathbb{Z}^n$ and describe Λ explicitly.
   (d) Let $\chi : C_n \to \{\pm 1\}$ be the unique character with $\chi(s_1) = 1$ and $\chi(s_2) = \chi(s_3) = \cdots = \chi(s_n) = -1$. Let G_3 be the subset of G consisting of pairs $(w, (a_1, \dots, a_n))$ where $\chi(w) = (-1)^{\sum a_j}$. Show that G_3 is a subgroup of G.
   (e) Construct a short exact sequence $1 \to \Lambda \to G_3 \to C_n \to 1$ where Λ is a free rank $\mathbb{Z}$-module with a C_n action, and show that this sequence is not semidirect. (Hint: Find an order 2 element in C_n that doesn't lift to an order 2 element of G_3.)

2. Let W be a finite crystallographic Coxeter group with a chosen crystallographic root system of rank n. Let $\tilde{W}$ be the affine Coxeter group $W \ltimes \mathbb{Z}\langle \alpha_i^\vee \rangle$. Recall that the affine hyperplane arrangement is the arrangement of hyperplanes $\{x \in V^\vee : \langle x, \beta \rangle = k\}$ for $\beta \in \Phi$ and $k \in \mathbb{Z}$.

   Let D be the fundamental domain in this hyperplane arrangement.
   (a) Show that every region of the hyperplane arrangement is of the form $wD + \gamma^\vee$ for precisely one $w \in W$ and $\gamma^\vee \in \mathbb{Z}\langle \alpha_i^\vee \rangle$.
   (b) Let Π be the parallelepiped $\{x \in V^\vee : 0 \leq \langle x, \alpha_i \rangle \leq 1\}$ for $1 \leq i \leq n$. Let M be the number of regions into which the affine hyperplane arrangement divides Π. Show that the index $[\mathbb{Z}\langle \alpha_i \rangle^\vee : \mathbb{Z}\langle \alpha_i^\vee \rangle]$ is equal to $\frac{|W|}{M}$.
   (c) The domain D is a simplex with walls $\{x \in V^\vee : \langle x, \alpha_i \rangle = 0\}$ for $1 \leq i \leq n$ and one additional wall $\{x \in V^\vee : \langle x, \theta \rangle = 1\}$ for some root θ. Let $\theta = \sum c_i \alpha_i$. Show that $M = \frac{\mathrm{Vol}(\Pi)}{\mathrm{Vol}(D)} = n! c_1 c_2 \cdots c_n$.
   (d) Multiplying the formulas above gives $|W| = n! c_1 \cdots c_n [\mathbb{Z}\langle \alpha_i \rangle^\vee : \mathbb{Z}\langle \alpha_i^\vee \rangle]$. Check this formula in type $\tilde{A}_n$.

3. For a Coxeter group W, put $W(q) = \sum_{w \in W} q^{\ell(w)}$. The combinatorics of this generating function is a fascinating topic we won't have time for, so we glimpse it here.

   In this problem, we derive a recursion for $W(q)$ in terms of parabolic subgroups. We recall the notation W_I for the parabolic subgroup and ${}^I W = \{w \in W : s_i \text{ is a left ascent of } w \text{ for all } i \in I\}$ from Problem Set 4. We write s_1, s_2, ..., s_n for the simple generators of W and $[n] = \{1, 2, \dots, n\}$.
   (a) Show that
   $$W_I(q) \cdot \left(\sum_{w \in {}^I W} q^{\ell(w)} \right) = W(q)$$
   (b) Show that
   $$\sum_{I \subseteq [n]} (-1)^{n-|I|} \sum_{w \in {}^I W} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & W \text{ finite} \\ 0 & W \text{ infinite} \end{cases}.$$
   (c) Combine the two parts above to give a recursion for $W(q)$ in terms of $W_I(q)$ for $I \subsetneq [n]$, and show that $W(q)$ is always a rational function of q.
   (d) Use the above formula to compute $\sum_{w \in \tilde{A}_2} q^{\ell(w)}$. You may take as known that $\sum_{w \in A_2} q^{\ell(w)} = 1 + 2q + 2q^2 + q^3$.