Problem Set 6: Due Friday, October 25

See the course website for homework policy.

1. Recall that a lattice is a partially ordered set P where any two elements x and y have a unique least upper bound $x \vee y$ and greatest lower bound $x \wedge y$. Verify that, in any lattice, the following equalities hold:
   (a) $\vee$ and $\wedge$ are commutative and associative.
   (b) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$.
   (c) We have $x \vee y = y$ if and only if $x \wedge y = x$

2. Let L be a set with binary operations $\vee$ and $\wedge$ which are commutative and associative and obey the relations $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. In this problem, we will show that L is a lattice.
   (a) Show that $x \vee x = x$ and $x \wedge x = x$.
   (b) Show that $x \vee y = y$ if and only if $x \wedge y = x$.
   (c) Define $x \preceq y$ if $x \vee y = y$. Show that $\preceq$ is a partial order.
   (d) Show that $\vee$ and $\wedge$ are the meet and join for this partial order.

3. In this question, you may assume that the type A weak order is a lattice.
   (a) Embed B_n into A_{2n-1} in the usual way. Show that B_n is closed under the lattice operations $\vee$ and $\wedge$ of A_n. Deduce that B_n is a lattice.
   (b) Embed D_n into B_n in the standard way. Show that D_n is **not** closed under the lattice operations of B_n.

4. The point of this question is to prove by hand that weak order on S_n is a lattice. Given $w \in S_n$, let G_w be the directed graph with vertex set $[n] := \{1, 2, \ldots, n\}$ and with an edge $i \to j$ if (ij) is an inversion of w, with $i < j$.

   Let u and $v \in S_n$. We define an antisymmetric relation $\prec_{uv}$ on $[n] = \{1, 2, \ldots, n\}$ as follows: For $i < j$ in S_n, put $i \succ_{uv} j$ if there is a directed path from i to j through $G_u \cup G_v$, and put $i \prec_{uv} j$ otherwise. Here $G_u \cup G_v$ is just the directed graph with vertex set $[n]$ whose edges are the union of G_u and G_v.
   (a) Show that $\prec_{uv}$ is acyclic relation. Let $u \cup v$ be the permutation whose inversion set is those (ij) with $i < j$ and $i \succ_{uv} j$.
   (b) Show that $u \cup v$ is the least upper bound of u and v.
   (c) Explain how similarly to find the greatest lower bound of u and v.