Problem Set 7: Due Friday, November 1

See the course website for homework policy.

1. Read the course notes from October 18 to November 30. Suggest something that can be improved, or a question they raise.

2. This question fills in the details of how permutahedra work. Take a Coxeter group and Cartan matrix as usual, and take roots and coroots that pair by the Cartan matrix such that both $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee)$ are linearly independent. Choose $\rho \in V$ such that $\langle \rho, \alpha_i^\vee \rangle > 0$ for all i (since the $\alpha_i^\vee$ are linearly independent, this is possible). Let $X = \{w\rho\}_{w \in W}$ and let P be the convex hull of X. The polytope P is called the W-permutahedron.

   (a) Show that $\rho - w\rho$ is a positive linear combination of $\{\beta_t : t \in \mathrm{inv}(w)\}$. Hint: Induction on $\ell(w)$.

   (b) Let $\theta \in V^\vee$, so $\langle \theta, \ \rangle$ is a linear function on P. Show that this function achieves its minimum at ρ, and nowhere else, if and only if $\theta \in D^\circ$. (Note that D° is nonempty since the α_i are linearly independent.)

   (c) Show that $\langle \theta, \ \rangle$ achieves its minimum at $w\rho$, and nowhere else, if and only if $\theta \in wD^\circ$.

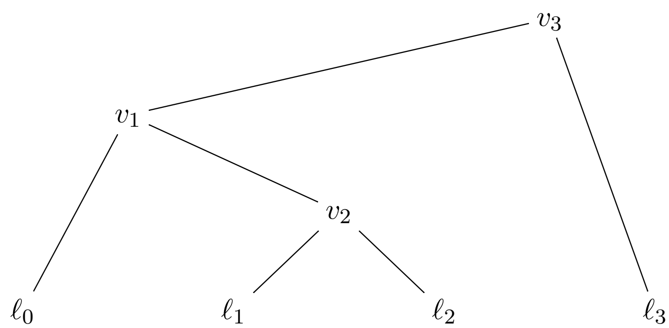
   (d) Let W_I be a standard finite parabolic subgroup of W and let D_I° be the corresponding face of D. Let P_I be the convex hull of $\{w\rho : w \in W_I\}$. Show that $\langle \theta, \ \rangle$ achieves its minimum on P_I and nowhere else if and only if $\theta \in D_I^\circ$. Analogously, show that $\langle \theta, \ \rangle$ achieves its maximum on uP_I and nowhere else if and only if $\theta \in uD_I^\circ$.

   (e) Take the standard root system for B_3, with simple roots e_1, $e_2 - e_1$ and $e_3 - e_2$. Sketch the three dimensional polytope P.

   (f) The previous example is for a finite Coxeter group, which is the usual context in which permutahedra are considered. But we can handle infinite groups. Consider the Cartan matrix $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$ with roots $\left[\begin{smallmatrix} 2 \\ 0 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 2 \\ 0 \end{smallmatrix}\right]$ and co-roots $\left[\begin{smallmatrix} 1 & -1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 & 1 & 1 \end{smallmatrix}\right]$. Take $\rho = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$. Show that X is the set of vectors of the form $\left[\begin{smallmatrix} -(k-1)k \\ -k(k+1) \\ 1 \end{smallmatrix}\right]$ for $k \in \mathbb{Z}$, and sketch the infinite "polytope" P.

The final question is on the back.

3. This question fills in the details of Loday's construction. Let T_n be the set of planar binary trees with leaves ℓ_0, ℓ_1, …, ℓ_n. We draw the leaves at the bottom of the tree from left to right. We number the internal vertices of the tree as v_1, v_2, …, v_n from left to right, so that v_k is to the right of ℓ_0, ℓ_1, …, ℓ_{k-1} and to the left of ℓ_k, ℓ_{k+1}, …, ℓ_n.



Given a tree T, define the integer vector $c(T)$ where $c(T)_i$ is the number of ordered pairs (j,k) such that ℓ_j is a left descendant of v_i and ℓ_k is a right descendant of v_i. In the example above, $c(T) = (2,1,3)$. Let Assoc_n be the convex hull of the vectors $c(T)$ for $T \in T_n$.

Given $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with the x_i distinct, let $T(\vec{x})$ be the unique binary tree which can be drawn such that v_i is at height x_i. We will first be showing that $\langle \vec{x}, \ \rangle$ is maximized on Assoc_n at the vertex $c(T(\vec{x}))$.

(a) Show that there is a continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that, if $\vec{x}$ is a vector with distinct entries, we have $\phi(\vec{x}) = \langle \vec{x}, c(T(\vec{x})) \rangle$.

(b) Let $\vec{x}$ and $\vec{y}$ be two vectors in $\mathbb{R}^n$ so that, for any $t \in \mathbb{R}$, the vector $t\vec{x} + (1-t)\vec{y}$ has at most two equal entries. Show that the restriction of ϕ to the line segment from $\vec{x}$ to $\vec{y}$ is convex. (Hint: This is a piecewise linear function; describe what happens at its corners.)

(c) Let $\vec{x}$ be a vector in $\mathbb{R}^n$ with distinct entries and let U be a tree other than $T(\vec{x})$. Show that $\langle \vec{x}, c(T(\vec{v})) \rangle > \langle \vec{x}, c(U) \rangle$. Hint: Choose a generic $\vec{y}$ such that $U = T(\vec{y})$ and consider what happens to ϕ on the line segment from $\vec{x}$ to $\vec{y}$.

(d) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. Show that $\langle \vec{x}, \ \rangle$, on Assoc_n, is maximized solely at $c(T(\vec{x}))$. This in particular shows that every $c(T)$ is a vertex of Assoc_n.

We now know that the normal fan to Assoc_n is given by coarsening the S_n hyperplane arrangement as described in class.

(e) Let T be a binary tree and let $\mathrm{Cone}(T)$ be the cone of all $\vec{x}$ with $T(\vec{x}) = T$. Show that $\mathrm{Cone}(T)$ and $\mathrm{Cone}(U)$ border along a codimension 1 face if and only if $c(T)$ and $c(U)$ differ by a single association.