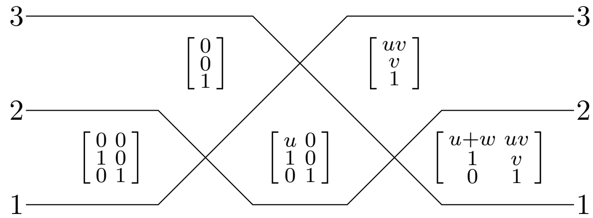
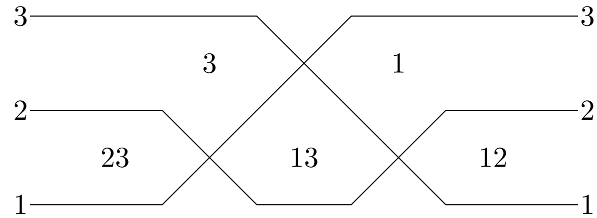
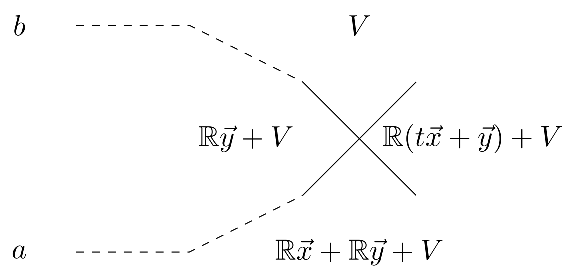
WORKSHEET 10: INVERTING THE UNIPOTENT PRODUCT II:

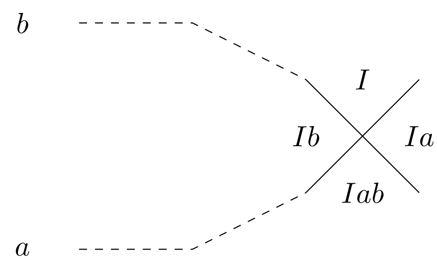
We recall our previous example:





Continue to assume that the word $s_{i_1} s_{i_2} \cdots s_{i_N}$ is reduced. We focus on a crossing j. Abbreviate i_j to i and t_j to t. Let the last $n - i_j + 1$ columns of g_{j-1} be $\begin{bmatrix} | & | & \\ \vec{x} & \vec{y} & V \\ | & | & \end{bmatrix}$, so the last $n - i_j + 1$ columns of g_j are $\begin{bmatrix} | & | & \\ \vec{x} & (t\vec{x}+\vec{y}) & V \\ | & | & \end{bmatrix}$. Let a be the source of the strand going up through the j-th crossing and let b be the source of the strand going down. So the chamber labels and the spaces look like:





Problem 10.1. Show that, as depicted in the picture, $a < b$.

The aim of the next few problems is to prove the formula:

$$t = \frac{\Delta^{Ib}\left(\begin{bmatrix}\vec{y} & V\end{bmatrix}\right)\ \Delta^{Ia}\left(\begin{bmatrix}t\vec{x}+\vec{y} & V\end{bmatrix}\right)}{\Delta^{Iab}\left(\begin{bmatrix}\vec{x} & \vec{y} & V\end{bmatrix}\right)\ \Delta^{I}\left(\begin{bmatrix}V\end{bmatrix}\right)}$$

where we have abused notation by writing V for a matrix whose columns are a basis of V.

Here we have written the argument of the minor as a matrix, not a subspace, so that the minor will be well-defined, not just defined up to scalars. We'll say more about that next time.

Problem 10.2. Show that $\Delta^{Ia}(\begin{bmatrix} y & V \end{bmatrix}) = 0$.

Problem 10.3. Show that $\Delta^{Ia}(\begin{bmatrix} t\vec{x}+\vec{y} & V \end{bmatrix}) = t\Delta^{Ia}(\begin{bmatrix} \vec{x} & V \end{bmatrix})$.

Problem 10.4. Show that

$$\Delta^{Iab}\left(\begin{bmatrix}\vec{x} & \vec{y} & V\end{bmatrix}\right)\ \Delta^{I}\left(\begin{bmatrix}V\end{bmatrix}\right) = \Delta^{Ib}\left(\begin{bmatrix}\vec{y} & V\end{bmatrix}\right)\ \Delta^{Ia}\left(\begin{bmatrix}\vec{x} & V\end{bmatrix}\right) - \Delta^{Ia}\left(\begin{bmatrix}\vec{y} & V\end{bmatrix}\right)\ \Delta^{Ib}\left(\begin{bmatrix}\vec{x} & V\end{bmatrix}\right).$$

(Remember your homework!)

Problem 10.5. Put all the parts together to prove

$$t = \frac{\Delta^{Ib}\left(\begin{bmatrix}\vec{y} & V\end{bmatrix}\right)\ \Delta^{Ia}\left(\begin{bmatrix}t\vec{x}+\vec{y} & V\end{bmatrix}\right)}{\Delta^{Iab}\left(\begin{bmatrix}\vec{x} & \vec{y} & V\end{bmatrix}\right)\ \Delta^{I}\left(\begin{bmatrix}V\end{bmatrix}\right)}$$