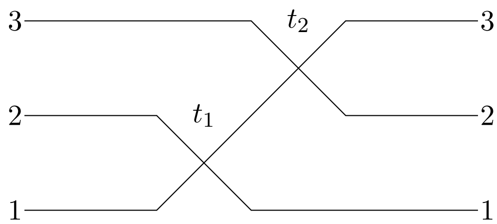
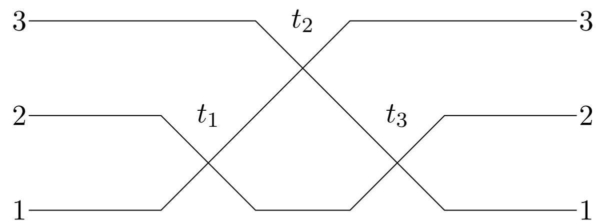
WORKSHEET 11: INVERTING THE UNIPOTENT PRODUCT III

Let $s_{i_1} s_{i_2} \cdots s_{i_N}$ be a reduced word in S_n. Let $t_1, t_2, \dots, t_N \in \mathbb{R}_{>0}$. Let

$$g_r = x_{i_1}(t_1) \cdots x_{i_r}(t_r),$$

this is a matrix in $N_+ \cap B_- w_r B_-$ where $w_r = s_{i_1} s_{i_2} \cdots s_{i_r}$. Let f_r be the unique matrix in $N_- w_r \cap w_r N_+$ with $g_r B_- = f_r B_-$. Here is a useful example:





$$g_2 = \begin{bmatrix} 1 & t_1 & t_1 t_2 \\ & 1 & t_2 \\ & & 1 \end{bmatrix} \quad f_2 = \begin{bmatrix} & & 1 \\ 1 & & t_1^{-1} \\ & 1 & t_1^{-1} t_2^{-1} \end{bmatrix} \qquad g_3 = \begin{bmatrix} 1 & t_1 + t_3 & t_1 t_2 \\ & 1 & t_2 \\ & & 1 \end{bmatrix} \quad f_3 = \begin{bmatrix} & & 1 \\ & 1 & t_1^{-1} \\ 1 & (t_1 + t_3) t_1^{-1} t_2^{-1} & t_1^{-1} t_2^{-1} \end{bmatrix}$$

Problem 11.1. Show that the $(i_r + 1)$-th column of f_{r-1} is the i_r-th column of f_r. For $k \neq i, i+1$, show that the k-th columns of f_r and f_{r-1} are the same.

Problem 11.2. Show that the span of the columns $i_r + 1, i_r + 2, i_r + 3, \dots, n$ is the same as the span of the columns $i_r, i_r + 2, i_r + 3, \dots, n$ of f_r. For $k \neq i_r + 1$, show that the span of columns $k, k+1, \dots, n$ of f_{r-1} is the same as the span of columns $k, k+1, \dots, n$ of f_r.

Problem 11.3. More generally, let $q < r$. Show that the span of columns $\{k, k+1, \dots, n\}$ of f_q is the same as the span of the columns $\{k, k+1, \dots, n\} s_{i_{q+1}} s_{i_{q+1}} \cdots s_{i_r}$ of f_r. Here we are using the standard action of S_n on $\{1, 2, \dots, n\}$, but as a right action.

Problem 11.4. Previously, we labeled chambers with the ratio of the top nonzero minor to the bottom nonzero minor for the rightmost columns of g_r. Explain how to adapt this to give a formula in terms solely of f_N.

Here is a larger example. In GL_4, we take the factorization $x_1(t_1) x_2(t_2) x_1(t_3) x_3(t_4) x_2(t_5) x_1(t_6)$. Here are the partial products:

$$\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&t_1&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&t_1&t_1t_2&0 \\ 0&1&t_2&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&t_1+t_3&t_1t_2&0 \\ 0&1&t_2&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&t_1+t_3&t_1t_2&t_1t_2t_4 \\ 0&1&t_2&t_2t_4 \\ 0&0&1&t_4 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&t_1+t_3&t_1t_2+t_5(t_1+t_3)&t_1t_2t_4 \\ 0&1&t_2+t_5&t_2t_4 \\ 0&0&1&t_4 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&t_1+t_3+t_6&t_1t_2+t_5(t_1+t_3)&t_1t_2t_4 \\ 0&1&t_2+t_5&t_2t_4 \\ 0&0&1&t_4 \\ 0&0&0&1 \end{bmatrix}$$

Here are the f-matrices:

$$\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&1&0&0 \\ 1&\frac{1}{t_1}&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&0&1&0 \\ 1&0&\frac{1}{t_1}&0 \\ 0&1&\frac{1}{t_1t_2}&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&0&1&0 \\ 0&1&\frac{1}{t_1}&0 \\ 1&\frac{t_1+t_3}{t_2t_3}&\frac{1}{t_1t_2}&0 \\ 0&0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&1 \\ 0&1&0&\frac{1}{t_1} \\ 1&\frac{t_1+t_3}{t_2t_3}&0&\frac{1}{t_1t_2} \\ 0&0&1&\frac{1}{t_1t_2t_4} \end{bmatrix} \begin{bmatrix} 0&0&0&1 \\ 0&0&1&\frac{1}{t_1} \\ 1&0&\frac{t_1+t_3}{t_2t_3}&\frac{1}{t_1t_2} \\ 0&1&\frac{t_1t_2+t_1t_5+t_3t_5}{t_2t_3t_4t_5}&\frac{1}{t_1t_2t_4} \end{bmatrix} \begin{bmatrix} 0&0&0&1 \\ 0&0&1&\frac{1}{t_1} \\ 0&1&\frac{t_1+t_3}{t_2t_3}&\frac{1}{t_1t_2} \\ 1&\frac{t_2t_3+t_2t_6+t_5t_6}{t_4t_5t_6}&\frac{t_1t_2+t_1t_5+t_3t_5}{t_2t_3t_4t_5}&\frac{1}{t_1t_2t_4} \end{bmatrix}$$