WORKSHEET 12: PRODUCTS OF CHEVALLEY GENERATORS IN GL_n

We write $x_j(t)$ for the matrix which has t in position $(j, j+1)$, which has 1's on the diagonal and 0's everywhere else. We write $y_j(t)$ for the matrix which has t in position $(j+1, j)$, which has 1's on the diagonal and 0's everywhere else. We write $\delta_j(t)$ for the diagonal matrix which is t in position (j, j) and has 1's in the other diagonal places.

We earlier proved:

$$x_i(\mathbb{R}_{>0})x_j(\mathbb{R}_{>0}) = x_j(\mathbb{R}_{>0})x_i(\mathbb{R}_{>0}) \text{ for } |i-j| \geq 2.$$
$$x_i(\mathbb{R}_{>0})x_{i+1}(\mathbb{R}_{>0})x_i(\mathbb{R}_{>0}) = x_{i+1}(\mathbb{R}_{>0})x_i(\mathbb{R}_{>0})x_{i+1}(\mathbb{R}_{>0}).$$
$$x_i(\mathbb{R}_{>0})x_i(\mathbb{R}_{>0}) = x_i(\mathbb{R}_{>0}).$$

Of course, we also have

$$y_i(\mathbb{R}_{>0})y_j(\mathbb{R}_{>0}) = y_j(\mathbb{R}_{>0})y_i(\mathbb{R}_{>0}) \text{ for } |i-j| \geq 2.$$
$$y_i(\mathbb{R}_{>0})y_{i+1}(\mathbb{R}_{>0})y_i(\mathbb{R}_{>0}) = y_{i+1}(\mathbb{R}_{>0})y_i(\mathbb{R}_{>0})y_{i+1}(\mathbb{R}_{>0}).$$
$$y_i(\mathbb{R}_{>0})y_i(\mathbb{R}_{>0}) = y_i(\mathbb{R}_{>0}).$$

Problem 12.1. Show that

$$\delta_i(\mathbb{R}_{>0})\delta_j(\mathbb{R}_{>0}) = \delta_j(\mathbb{R}_{>0})\delta_i(\mathbb{R}_{>0})$$
$$\delta_i(\mathbb{R}_{>0})x_j(\mathbb{R}_{>0}) = x_j(\mathbb{R}_{>0})\delta_i(\mathbb{R}_{>0})$$
$$\delta_i(\mathbb{R}_{>0})y_j(\mathbb{R}_{>0}) = y_j(\mathbb{R}_{>0})\delta_i(\mathbb{R}_{>0})$$
$$\delta_i(\mathbb{R}_{>0})\delta_i(\mathbb{R}_{>0}) = \delta_i(\mathbb{R}_{>0}).$$

Problem 12.2. Show that

$$x_i(\mathbb{R}_{>0})y_j(\mathbb{R}_{>0}) = y_j(\mathbb{R}_{>0})x_i(\mathbb{R}_{>0})$$

for $i \neq j$.

Problem 12.3. Show that

$$x_i(\mathbb{R}_{>0})y_i(\mathbb{R}_{>0})\delta_i(\mathbb{R}_{>0})\delta_{i+1}(\mathbb{R}_{>0}) = y_i(\mathbb{R}_{>0})x_i(\mathbb{R}_{>0})\delta_i(\mathbb{R}_{>0})\delta_{i+1}(\mathbb{R}_{>0}).$$

Problem 12.4. Consider any product where each term is of the form $x_i(\mathbb{R}_{>0})$, $y_j(\mathbb{R}_{>0})$, $\delta_k(\mathbb{R}_{>0})$ and where each of $\delta_1(\mathbb{R}_{>0})$, $\delta_2(\mathbb{R}_{>0})$, …, $\delta_n(\mathbb{R}_{>0})$ appears at least once. Let i_1, i_2, …, i_M be the sequence of subscripts of the x_i factors and let j_1, j_2, …, j_N be the sequence of subscripts of the y_j factors. Prove that the image of this product in $\mathrm{GL}_n(\mathbb{R})$ depends only on the 0-Hecke products $e_{i_1} * e_{i_2} * \cdots * e_{i_M}$ and $e_{j_1} * e_{j_2} * \cdots * e_{j_N}$.

Let u and v in S_n. We define $M^{u,v}$ to be the product in Problem 12.4, where $u = e_{i_1} * e_{i_2} * \cdots * e_{i_M}$ and $v = e_{j_1} * e_{j_2} * \cdots * e_{j_N}$.

Problem 12.5. Show that $M^{u,v} \subseteq B_- u B_- \cap B_+ v B_+$.

As you would guess, our eventual goal is that $M^{u,v}$ is the totally nonnegative part of $B_- u B_- \cap B_+ v B_+$ and, if $s_{i_1} \cdots s_{i_M}$ and $s_{j_1} \cdots s_{j_N}$ are reduced, then this gives a diffeomorphism $M^{u,v} \cong \mathbb{R}_{>0}^{n+\ell(u)+\ell(v)}$. This is a theorem of Fomin and Zelevinsky, "Double Bruhat Cells and Total Positivity", *JAMS*, Volume 12, Number 2, April 1999, Pages 335–380.

This is a good chance to prove a lemma which we'll need in the future:

Problem 12.6. Show that there is a continuous (in fact, polynomial) function $g : \mathbb{R}_{\geq 0} \to \mathrm{GL}_n(\mathbb{R})$ such that $g(t)$ is totally positive for $t > 0$ and $g(0) = Id_n$.