WORKSHEET 13: LDU FACTORIZATION

We recall the reader of the following notational conventions: $[k] = \{1, 2, \ldots, k\}$. The minor $\Delta^R_C(X)$ uses row set R and column set C.

An ***LDU factorization*** of an $n \times n$ matrix X is a factorization $X = LDU$ where $L \in N_-$, $U \in N_+$ and D is diagonal. In this course, we will always require that D is diagonal and **invertible**.

Problem 13.1. Show that, if X has an LDU factorization, then the minors $\Delta^{[k]}_{[k]}(X)$ are nonzero for $1 \leq k \leq n$.

Problem 13.2. Show the converse of Problem 13.1: If X is an $n \times n$ matrix and $\Delta^{[k]}_{[k]}(X) \neq 0$ for $1 \leq k \leq n$, then X has an LDU-factorization.

Problem 13.3. Let $X = LDU$ be an LDU factorization. Show that the entries of L, D and U are given by the following formulas:

$$L_{ij} = \frac{\Delta^{[j-i]\cup\{i\}}_{[j]}(X)}{\Delta^{[j]}_{[j]}(X)} \text{ for } i > j \qquad U_{ij} = \frac{\Delta^{[i]}_{[i-1]\cup\{j\}}(X)}{\Delta^{[i]}_{[i]}(X)} \text{ for } i < j \qquad D_{jj} = \frac{\Delta^{[j]}_{[j]}(X)}{\Delta^{[j-1]}_{[j-1]}(X)}.$$

In particular, deduce that L, D and U are uniquely determined by X.