WORKSHEET 14: LDU FACTORIZATION AND TOTAL POSITIVITY

Our goal is to show

Theorem. Let X be an invertible $n \times n$ totally nonnegative matrix. Then X has an LDU factorization where L, D and U are totally nonnegative.

This result is due to Cryer: "The LU-factorization of totally positive matrices", *Linear Algebra and its Applications*, Volume 7, Issue 1, January 1973, Pages 83–92, and our proof is close to his.

We first set out to show that X has an LDU factorization. Since X is invertible, $\Delta_{[n]}^{[n]}(X) \neq 0$. Suppose, for the sake of contradiction, that there is some index m for which $\Delta_{[m-1]}^{[m-1]}(X) = 0$ but $\Delta^{[m]}[m](X) \neq 0$.

Problem 14.1. Show that the upper left $(m-1) \times m$ and $m \times (m-1)$ submatrices of X must have rank $m-1$.

Problem 14.2. Show that there are some p and q in $[m-1]$ such that $\Delta_{[m-1]}^{[m]\setminus\{p\}}(X)$ and $\Delta_{[m]\setminus q}^{[m-1]}(X)$ are nonzero.

Problem 14.3. Obtain a contradiction by looking at the identity:

$$\Delta_{[m]}^{[m]}(X)\Delta_{[m-1]\setminus\{q\}}^{[m-1]\setminus\{p\}}(X) = \Delta_{[m-1]}^{[m-1]}(X)\Delta_{[m]\setminus\{q\}}^{[m]\setminus\{p\}}(X) - \Delta_{[m-1]}^{[m]\setminus\{p\}}(X)\Delta_{[m]\setminus\{q\}}^{[m-1]}(X).$$

We now know that X has an LDU factorization, say $X = LDU$. We now turn to showing that L, D and U are totally nonnegative.

Problem 14.4. Show that D is totally nonnegative. (This one is easy.)

We next consider the special case that X is **totally positive**. This was on the problem sets, but I'd like to make sure everyone knows how to do it.

Problem 14.5. With the extra hypothesis that X is **totally positive**, show that all the left justified minors of L, and all the top justified minors of U, are positive. Conclude that L and U are totally positive in the sense that we used for upper triangular matrices, meaning that $\Delta_{j_1 j_2 \cdots j_k}^{i_1 i_2 \cdots i_k}(U) > 0$ if $i_t \leq j_t$ for all t, and $\Delta_{j_1 j_2 \cdots j_k}^{i_1 i_2 \cdots i_k}(L) > 0$ if $i_t \geq j_t$ for all t.

We now return to assuming that X is totally nonnegative and invertible. In Problem 12.6, you showed that there is a continuous (in fact, polynomial) function $g : \mathbb{R}_{\geq 0} \to \mathrm{GL}_n(\mathbb{R})$ such that $g(t)$ is totally positive for $t > 0$ and $g(0) = Id_n$.

Problem 14.6. Show that $g(t)Xg(t)$ is totally positive for $t > 0$.

Problem 14.7. Use a limiting argument to prove the Theorem: $X = LDU$ with L, D and U totally nonnegative.