# WORKSHEET 15: LDU FACTORIZATION AND DOUBLE BRUHAT CELLS

We now know that, if $X$ is a totally nonnegative, invertible, $n \times n$ matrix then $X$ has a unique $LDU$ factorization where $L$ is a totally nonnegative element of $N_-$, $D$ is a positive diagonal matrix and $U$ is a totally nonnegative element of $N_+$.

Intersecting two Bruhat decompositions of $\mathrm{GL}_n$, we obtain the decomposition of $GL_n$ into ***double Bruhat cells***:

$$\mathrm{GL}_n = \bigsqcup_{u,v} (B_+ u B_+ \cap B_- v B_-) .$$

**Problem 15.1.** Show that, if $X \in B_+ u B_+ \cap B_- v B_-$, then $L \in B_+ u B_+ \cap N_-$ and $U \in B_- v B_- \cap N_+$.

**Problem 15.2.** Show that $(L, D, U) \mapsto LDU$ is a diffeomorphism

$$(B_+ u B_+ \cap N_-)_{\geq 0} \times \mathbb{R}^n_{>0} \times (B_- v B_- \cap N_+)_{\geq 0} \longrightarrow (B_+ u B_+ \cap B_- v B_-)_{\geq 0}.$$

Here the subscript $\geq 0$ indicates "totally nonnegative" in each case.

Consider any product where each term is of the form $x_i(\mathbb{R}_{>0})$, $y_j(\mathbb{R}_{>0})$, $\delta_k(\mathbb{R}_{>0})$ and where each of $\delta_1(\mathbb{R}_{>0})$, $\delta_2(\mathbb{R}_{>0})$, ..., $\delta_n(\mathbb{R}_{>0})$ appears at least once. Let $i_1$, $i_2$, ..., $i_M$ be the sequence of subscripts of the $x_i$ factors and let $j_1$, $j_2$, ..., $j_N$ be the sequence of subscripts of the $y_j$ factors. Suppose that $s_{i_1} s_{i_2} \cdots s_{i_M}$ is a reduced word for $u$ and that $s_{j_1} s_{j_2} \cdots s_{j_N}$ is a reduced word for $v$.

Prove the following result of Fomin and Zelevinsky:

**Problem 15.3.** Show that the product gives a diffeomorphism $\mathbb{R}^{\ell(u)+\ell(v)+n}_{>0} \longrightarrow (B_+ u B_+ \cap B_- v B_-)_{\geq 0}$.