WORKSHEET 16: KASTELEYN LABELINGS

Let G be a bipartite graph, with black vertex set B and white vertex set W. For now, let us assume that $\#(B) = \#(W)$. A ***matching*** of G is a set of edges which cover each vertex of G exactly once. A matching is also called a ***perfect matching*** or a ***dimer cover***.

Let $x(e)$ be formal variables, indexed by the edges e of G. For a matching M, let $x(M) = \prod_{e \in M} x(e)$. Define the ***weighted adjacency matrix*** of G to be the matrix with rows indexed by B and columns indexed by W, where A_{bw} is $x(e)$ if $e = (b, w)$ and A_{bw} is 0 if there is no edge from b to w. Notice that we have

$$\det A = \sum_M \pm x(M).$$

We would like to get rid of the plus and minus signs. We would therefore like to replace each $x(e)$ by $\kappa(e)x(e)$ where $\kappa(e) \in \mathbb{C}^\times$ is some function on the edges which will make the signs cancel.

We define κ to be a ***Kasteleyn labeling*** if the following condition is met: Let C be a cycle of G with edges e_1, e_2, $\ldots$, e_{2k-1}, e_{2k}, Suppose that $G \setminus C$ has a matching. Then we impose that

$$(-1)^{k-1}\kappa(e_1)\kappa(e_3)\cdots\kappa(e_{2k-1}) = \kappa(e_2)\kappa(e_4)\cdots\kappa(e_{2k}). \qquad (*)$$

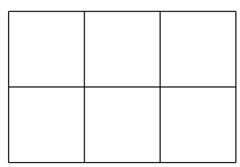
For future use, we will make this definition even if $\#(B) \neq \#(W)$.

Problem 16.1. Let κ be a Kasteleyn labeling of G. Let A^κ be the matrix formed by replacing $x(e)$ by $\kappa(e)x(e)$ in A. Show that

$$\det A^\kappa = c \sum_M x(M)$$

for some scalar c.

Problem 16.2. Find a Kasteleyn labeling of



It's okay if you don't check every single cycle; I just want you to have a bit of experience before we prove general results.

Problem 16.3. In the above example, give an example of a cycle C such that $G \setminus C$ does **not** have a matching. Does condition $(*)$ hold?

Problem 16.4. Let G be a planar graph where every interior face is a disk and let κ be a function from the edges of G to $\mathbb{C}^\times$. Suppose that condition $(*)$ holds whenever e_1, e_2, $\ldots$, e_{2k-1}, e_{2k}, is the boundary of a face of G. Show that κ is a Kasteleyn labeling.

Problem 16.5. Show that every planar bipartite graph has a Kasteleyn labelleing. (If you like, you may restrict to the case that every face bounds a disk, although you don't need to.)