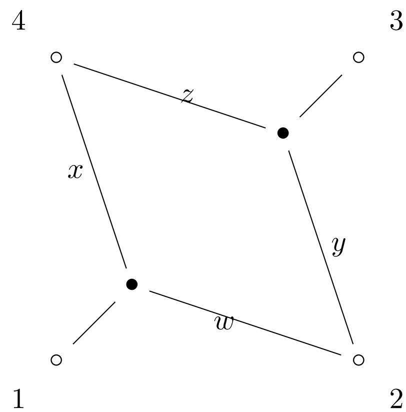
So far, we have been assuming that $\#(B) = \#(W)$. Now suppose instead that G is embedded in a disc D such that all the vertices on $\partial(D)$ are white. We'll write $\partial(G)$ for these white boundary vertices, W_0 for the interior white vertices and B for the black vertices. Let $\#(W_0) = m$, $\#(B) = m + k$ and $\#(\partial(G)) = n$, and fix a labeling of $\partial(G)$ by $[n]$ in counterclockwise order.

We'll define an ***almost perfect matching*** M to be a collection of edges which covers every vertex in $B \cup W_0$ exactly once, and every vertex in $\partial(G)$ at most once. For an almost perfect matching M, define $\partial(M)$ to be the set of edges in $\partial(G)$ which are covered by M. For a k-element subset I of $[n]$, define

$$\mu_I(G) = \sum_{\partial(M)=I} x(M).$$

We call the μ_I the ***boundary measurements*** of G. To avoid silly boundary cases, assume that G has at least one almost perfect matching.

Problem 17.1. Compute the μ_I for the graph below. The letters written on the edges are the $x(e)$ values; unlabeled edges have $x(e) = 1$.



Problem 17.2. Let A be the $(m + k) \times (m + n)$ adjacency matrix of G. Show that there is a Kasteleyn labeling κ of G such that $\Delta^B_{W_0 \cup I}(A^\gamma) = \mu_I(G)$.

Problem 17.3. Give an example of such a matrix in the case above.

Problem 17.4. Show that there is a $k \times n$ matrix X such that $\Delta^{[k]}_I(X) = \Delta^B_{W_0 \cup I}(A^\kappa)$. Conclude that (μ_I) is a point of the Grassmannian and, if all the $x(e)$ are nonnegative, it is a point of the Grassmannian with nonnegative Plucker coordinates.

We have now constructed, for any planar graph G as above, a map

$$\mu_G : \mathbb{R}^{\mathrm{Edges}(G)}_{>0} \to G(k, n)_{\geq 0}.$$

Problem 17.5. On the problem sets, you described points of $G(2, 4)_{\geq 0}$ with the following sign patterns. Give graphs that realize them:

$$\Delta^{12}, \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24}, \Delta^{34} > 0$$
$$\Delta^{12}, \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24} > 0, \quad \Delta^{34} = 0$$
$$\Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24} > 0, \quad \Delta^{12} = \Delta^{34} = 0$$
$$\Delta^{12}, \Delta^{13}, \Delta^{14} > 0 \quad \Delta^{23} = \Delta^{24} = \Delta^{34} = 0$$
$$\Delta^{12}, \Delta^{13}, \Delta^{23} > 0 \quad \Delta^{14} = \Delta^{24} = \Delta^{34} = 0$$