WORKSHEET 18: CYCLIC RANK MATRICES AND RELATED CONCEPTS

We have now described how we will parametrize portions of the totally nonnegative Grassmannian, using dimer covers. Our constructions will have a cyclic symmetry that is, perhaps, surprising at first, so we begin by explaining this:

Problem 18.1. Let M be a $k \times n$ matrix of rank k. Let σ be the permutation of $[n]$ with $\sigma(i) \equiv i + 1 \bmod n$. Construct a matrix M' with $\Delta^I(M') = \Delta^{\sigma(I)}(M)$ for all k-element subsets I of $[n]$. (Hint: The solutions will look slightly different for k even and k odd.)

Let M be a $k \times n$ matrix of rank k. For any $i \in \mathbb{Z}$, let M_i be the column of M whose position is $i \bmod n$. For $a \leq b$, define

$$r_{ab}(M) = \operatorname{rank}(M_a, M_{a+1}, M_{a+2}, \ldots, M_b).$$

It is also convenient to put $r_{i(i-1)}(M) = 0$ and $r_{i(i-2)}(M) = -1$. (More generally, it is convenient to put $r_{ij} = -i + j + 1$ for $i > j$.)

Problem 18.2. Show that, for any M, the matrix $r_{ij}(M)$ has the following properties:

(1) $r_{(i+1)j} \leq r_{ij} \leq r_{(i+1)j} + 1$ and $r_{i(j-1)} \leq r_{ij} \leq r_{i(j-1)} + 1$.
(2) If $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$ then $r_{ij} = r_{(i+1)(j-1)}$.
(3) $r_{ij} = k$ if $j \geq i + n - 1$.
(4) $r_{ij} = r_{(i+n)(j+n)}$

We define a matrix r_{ij} obeying the conditions of problem 18.2 (for some parameters (k, n)) to be a ***cyclic rank matrix***. Just as we bijected rank matrices with permutations, we want to find a similar object to index cyclic rank matrices.

Problem 18.3. Let r be a cyclic rank matrix. For each $i \in \mathbb{Z}$, show that there is a unique index $f(i)$ such that

$$r_{i\ f(i)} = r_{(i+1)\ f(i)} = r_{i\ (f(i)-1)} = r_{(i+1)\ (f(i)-1)} + 1.$$

Problem 18.4. Let r be a cyclic rank matrix and let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined in the previous problem. Show that

(1) $f : \mathbb{Z} \to \mathbb{Z}$ is a bijection.
(2) $f(i + n) = f(i) + n$.
(3) $i \leq f(i) \leq i + n$.
(4) $\frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = k$.

We will define a function f obeying these conditions to be a ***bounded affine permutation of type*** (k, n).

Remark: If we consider conditions 1, 2 and replace condition 4 by $\frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = 0$, this would be the so-called ***type*** $\tilde{A}$ ***Coxeter group***. If we consider just conditions 1, 2, this is sometimes called the ***affine symmetric group***.

Problem 18.5. Show that cyclic rank matrices are in bijection with bounded affine permutations (for the same (k, n)).