WORKSHEET 19: MORE ON CYCLIC RANK MATRICES

Recall that a ***cyclic rank matrix of type*** (k, n) is an array of integers r_{ij}, indexed by i and $j \in \mathbb{Z}$, obeying

(1) $r_{(i+1)j} \leq r_{ij} \leq r_{(i+1)j} + 1$ and $r_{i(j-1)} \leq r_{ij} \leq r_{i(j-1)} + 1$.
(2) If $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$ then $r_{ij} = r_{(i+1)(j-1)}$.
(3) $r_{ij} = k$ if $j \geq i + n - 1$.
(4) $r_{ij} = -i + j + 1$ if $i > j$.
(5) $r_{ij} = r_{(i+n)(j+n)}$.

A ***bounded affine permutation of type*** (k, n) is a map $f : \mathbb{Z} \to \mathbb{Z}$ such that

(1) $f : \mathbb{Z} \to \mathbb{Z}$ is a bijection.
(2) $f(i + n) = f(i) + n$.
(3) $i \leq f(i) \leq i + n$.
(4) $\frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = k$.

We saw last time that bounded affine permutations are in bijection with cyclic rank matrices. Specifically, $f(i) = j$ if and only if $r_{ij} = r_{(i+1)j} = r_{i(j-1)} = r_{(i+1)(j-1)} + 1$.

We also constructed a map from full rank $k \times n$ matrices to cyclic rank matrices: If the matrix M has columns M_1, M_2, ..., M_n, then, for $i \leq j$, we take $r_{ij} = \operatorname{rank}(M_i, M_{i+1}, \ldots, M_j)$. (For $i > j$, we take $r_{ij} = -i + j + 1$).

Problem 19.1. Let M be a full rank $k \times n$ matrix with corresponding cyclic rank matrix r and bounded affine permutation f. Show that

(1) We have $f(i) = i$ if and only if $M_i = 0$.
(2) We have $f(i) = i + 1$ if and only if M_i and M_{i+1} are parallel, nonzero, vectors.
(3) We have $f(i) = i + n$ if and only if M_i is not in the span of M_{i+1}, M_{i+2}, ..., M_{i+n-1}.

Problem 19.2. With notation as above, show that $f(i) = j$ if and only if $M_i \in \operatorname{Span}(M_{i+1}, \ldots, M_{j-1}, M_j)$ and $M_i \notin \operatorname{Span}(M_{i+1}, \ldots, M_{j-1})$. (This observation was made to me by Allen Knutson.)

Let r be a cyclic rank matrix. For $i \in \mathbb{Z}$, let $\tilde{I}_i = \{j \geq i \ : \ r_{ij} > r_{i(j-1)}\}$. Let I_i be the reduction of $\tilde{I}_i$ modulo n. Let I_i be the reduction of $\tilde{I}_i$ to $\mathbb{Z}/n\mathbb{Z}$. We note that $I_{i+n} = I_i$, so we can consider the subscripts of the I_i to be cyclic modulo n.

Problem 19.3. Show that we can reconstruct f from I_i by the following recipe:

(1) If $I_i \neq I_{i+1}$, then $f(i)$ is determined by the conditions that $I_{i+1} \setminus I_i = \{f(i)\}$ and $i < f(i) < i + n$.
(2) If $I_i = I_{i+1}$ and $i \in I_i$, then $f(i) = i + n$.
(3) If $I_i = I_{i+1}$ and $i \notin I_i$, then $f(i) = i$.

Problem 19.4. Define a ***Grassmann necklace of type*** (k, n) to be a sequence $(I_1, I_2, \ldots, I_n)$ of k-element subsets of $[n]$ such that, for each i, we have $I_i \setminus \{i\} \subseteq I_{i+1}$. Show that Grassmann necklaces are in bijection with bounded affine permutations of type (k, n).