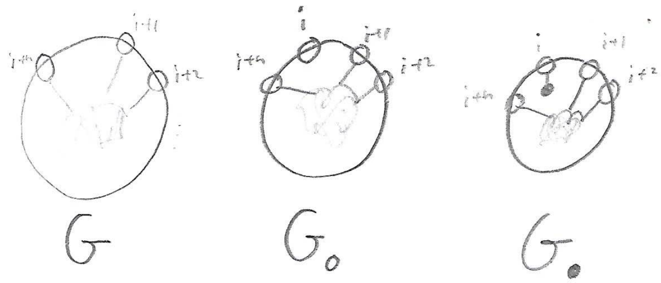
Worksheet 20: Lollipops

Let G be a planar bipartite graph, embedded in a disc D, such that all the vertices on $\partial(D)$ are white. We'll write $\partial(G)$ for these white boundary vertices, W_0 for the interior white vertices and B for the black vertices. Let $\#(W_0) = m$, $\#(B) = m + k$ and $\#(\partial(G)) = n$, and fix a labeling of $\partial(G)$ by $[n]$ in counterclockwise order.

Let $G_\circ$ and $G_\bullet$ be the graphs obtained from G as shown in the figures below. We number the new vertex i and fit the other numbers around it as shown.



This operation is sometimes known as ***adding a lollipop***.

So $G_\circ$ and $G_\bullet$ have types $(k, n+1)$ and $(k+1, n+1)$. Let f, $f_\circ$ and $f_\bullet$ be the corresponding affine permutations.

Problem 20.1. Show that $f_\circ(i) = i$ and $f_\bullet(i) = i + (n+1)$. Show that, for $i + 1 \leq j \leq i + n$, we have $f_\circ(j) = f_\bullet(j) = f(j)$.

Problem 20.2. Let f, $f_\circ$ and $f_\bullet$ be bounded affine permutations as above. Show that there are isomorphisms between the regions of $G(k, n)$, $G(k, n+1)$ and $G(k+1, n+1)$ with bounded affine permutations f, $f_\circ$ and $f_\bullet$. More specifically, if$M$, $M_\circ$ and $M_\bullet$ are corresponding matrices, we should have $\Delta^I(M) = \Delta^I(M_\circ) = \Delta^{I \cup \{i\}}(M_\bullet)$ for $I \subseteq \{i+1, \ldots, i+n\}$; we should have $\Delta^J(M_\circ) = 0$ if $i \in J$ and $\Delta^K(M_\bullet) = 0$ if $i \notin K$.