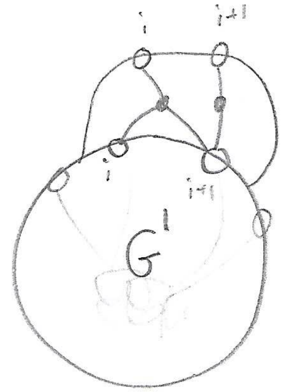
WORKSHEET 21: BRIDGES AND CHEVALLEY GENERATORS

Let G' be a planar bipartite graph, embedded in a disc D, such that all the vertices on $\partial(D)$ are white. We'll write $\partial(G')$ for these white boundary vertices, W_0 for the interior white vertices and B for the black vertices. Let $\#(W_0) = m$, $\#(B) = m + k$ and $\#(\partial(G)) = n$, and fix a labeling of $\partial(G')$ by $[n]$ in counterclockwise order.

Create a new graph G as shown in the figure below; the unlabeled edges have weight 1.



This operation is sometimes called ***adding a bridge***.

Problem 21.1. Let I be a k-element subset of $[n]$. Show that the boundary measurements of G and G' obey:

(1) If $i+1 \notin I$, then $\mu^I(G) = \mu^I(G')$.
(2) If $i+1 \in I$, then $\mu^I(G) = \mu^I(G') + t\mu^{I\setminus\{i+1\}\cup\{i\}}(G')$.

Let $x_i(t)$ be the matrix in GL_n with 1's on the diagonal, t in position $(i, i+1) \bmod n$ and 0's everywhere else.

Problem 21.2. With notation as above, let M' be a $k \times n$ matrix with $\Delta^I(M') = \mu^I(G')$. Let $M = M'x_i(t)$. Show that $\Delta^I(M) = \mu^I(G)$.

We now study the effect on cyclic rank matrices and associated objects. Let G' and G, M' and M be as above. Let r', f' and I' be the cyclic rank matrix, bounded affine permutation and Grassmann necklace of M', and likewise for M.

Problem 21.3. Suppose that $f'(i) > f'(i+1)$. In the above notation, show that $I_{i+1} = I'_{i+1} \setminus \{f(i)\} \cup \{f(i+1)\}$ and $I_j = I'_j$ for $j \neq i+1$. Show that $f(i) = f'(i+1)$, $f(i+1) = f'(i)$ and $f'(j) = f(j)$ for $j \not\equiv i,\ i+1 \bmod n$. You'll probably want to look back to Problem 19.2.

We are now ready to show that every bounded affine permutation occurs for some positive matrix. Let f be a bounded affine permutation of type (k, n).

Problem 21.4. Explain why we are done if there is some index i with $f(i) = i$ or $f(i) = i + n$.

Now, assume that $i < f(i) < i + n$ for all i.

Problem 21.5. Show that there is some index i with $f(i) < f(i+1)$.

Define

$$f'(j) = \begin{cases} f(j+1) & j \equiv i \bmod n \\ f(j-1) & j \equiv i+1 \bmod n \\ f(j) & j \not\equiv i,\ i+1 \bmod n \end{cases}.$$

Problem 21.6. Show that f' is a bounded affine permutation.

Problem 21.7. Assume inductively that there is a planar graph G' corresponding to the bounded affine permutation f'. Show that there is a graph G corresponding to f.