WORKSHEET 22: EVERY TOTALLY NONNEGATIVE MATRIX COMES FROM A PLANAR GRAPH

We are ready for our final result. That every totally nonnegative matrix M comes from a graph G.

Let f be the bounded affine permutation associated to M. We already showed (Problem 20.2) that we can reduce to a smaller Grassmannian if there is any i with $f(i) = i$ or $f(i) = i + n$, so assume there is not.

We have also already shown that there is an index i with $f(i) < f(i + 1)$. Define

$$f'(j) = \begin{cases} f(j+1) & j \equiv i \bmod n \\ f(j-1) & j \equiv i+1 \bmod n \\ f(j) & j \not\equiv i,\ i+1 \bmod n \end{cases}.$$

Suppose, inductively, we know that every totally nonnegative matrix M' with bounded affine permutation f' comes from a graph G'.

Recall that $x_i(t)$ is the matrix in GL_n with 1's on the diagonal, t in position $(i, i + 1) \bmod n$ and 0's everywhere else.

The rest of this worksheet is closely based on lecture notes by Thomas Lam[1], particularly Section 3.3.

Problem 22.1. We first consider the case where $f(i) = i + 1$. Let M' be the matrix with $M'_j = M_j$ for $j \neq i + 1$ and $M'_{i+1} = 0$. Show that there is a positive scalar t such that $M = M' x_i(t)$. Explain how to finish the proof in this case.

From now on, assume that $f(i) > i + 1$. Put $f(i) = a$ and $f(i + 1) = b$, so $i + 1 < a < b$.

Problem 22.2. Show that there is a $k-2$ element subset R of $[n]$ such that $I_i = R \cup \{i, i+1\}$, $I_{i+1} = R \cup \{i+1, a\}$, $I_{i+2} = R \cup \{a, b\}$.

Problem 22.3. Using the Plücker relation

$$\Delta^{R\cup\{i,a\}} \Delta^{R\cup\{i+1,b\}} = \Delta^{R\cup\{i,b\}} \Delta^{R\cup\{i+1,a\}} + \Delta^{R\cup\{i,i+1\}} \Delta^{R\cup\{a,b\}}$$

deduce that $\Delta^{R\cup\{i,a\}}(M) > 0$.

It will now be convenient to put $R = S \cup \{a\}$. So $I_{i+1} = S \cup \{i + 1\}$ and we just showed that $\Delta^{S\cup\{i\}}(M) > 0$. Put $t = \frac{\Delta^{S\cup\{i+1\}}(M)}{\Delta^{S\cup\{i\}}(M)}$ and put $M' = M x_i(-t)$.

Problem 22.4. Show that $t > 0$.

Problem 22.5. Let I be a k-element subset of $[n]$. Show that

$$\Delta^I(M') = \begin{cases} \Delta^I(M) & i+1 \notin I \\ \Delta^I(M) - t\Delta^{I\setminus\{i+1\}\cup\{i\}}(M) & i+1 \in I \end{cases}$$

Problem 22.6. Show that the bounded affine permutation of M' is f'.

The last remaining task is to show that M' is totally nonnegative.

Problem 22.7. Show that, if $i + 1 \notin I$, then $\Delta^I(M') \geq 0$.

Now, suppose that $I = T \sqcup \{i + 1\}$. From your homework,

$$\Delta^{S\cup\{i\}}(M)\Delta^{T\cup\{i+1\}}(M) \geq \Delta^{S\cup\{i+1\}}(M)\Delta^{T\cup\{i\}}(M).$$

Problem 22.8. Explain why this shows that $\Delta^I(M') \geq 0$.

We have now shown that every totally nonnegative point of $G(k, n)$ comes from a planar graph.

Problem 22.9. Make sure you understand why.

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[1] `http://www.math.lsa.umich.edu/~tfylam/Math665a/positroidnotes.pdf`