WORKSHEET 2: THE GESSEL-LINDSTRÖM-VIENNOT LEMMA

Let G be a finite directed acyclic graph, with a weight $w(e)$ assigned to each edge e. Given a path γ of the form $\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \cdots \xrightarrow{e_k} \bullet$, the weight $w(\gamma)$ of this path is defined to be $w(e_1)w(e_2)\cdots w(e_k)$. Given two subsets S and T of the vertices, we define a matrix M with rows indexed by S and columns indexed by T, so that

$$M_{st} = \sum_{s \overset{\gamma}{\rightsquigarrow} t} w(\gamma).$$

Theorem (Gessel-Lindström-Viennot: Let $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$. Suppose that G is a planar graph and that S and T occur on the boundary of the outer face, in order s_1, s_2, …, s_k, t_k, …, t_2, t_1. Then

$$\Delta^I_J(M) = \sum\nolimits_{s_1 \overset{\gamma_1}{\rightsquigarrow} t_1,\ s_2 \overset{\gamma_2}{\rightsquigarrow} t_2,\ \cdots,\ s_k \overset{\gamma_k}{\rightsquigarrow} t_k} w(\gamma_1)w(\gamma_2)\cdots w(\gamma_k)$$

where the sum is over k-tuples of vertex disjoint paths $(\gamma_1, \gamma_2, \cdots, \gamma_k)$ where γ_r is a path from s_r to t_r.

Can you give graphs G such that, as the weights on the edges vary over $\mathbb{R}_{>0}$, we parametrize:

$$\{\left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right] : w, x, y, z > 0,\ wz - xy > 0\}$$

$$\{\left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right] : w, x, y, z > 0,\ wz - xy = 0\}$$

$$\left\{\left[\begin{smallmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right] : x, y > 0\right\}$$

$$\left\{\left[\begin{smallmatrix} 1 & x & xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right] : x, y > 0\right\}$$