WORKSHEET 3: PARAMETRIZATION OF TOTALLY POSITIVE UNIPOTENT MATRICES

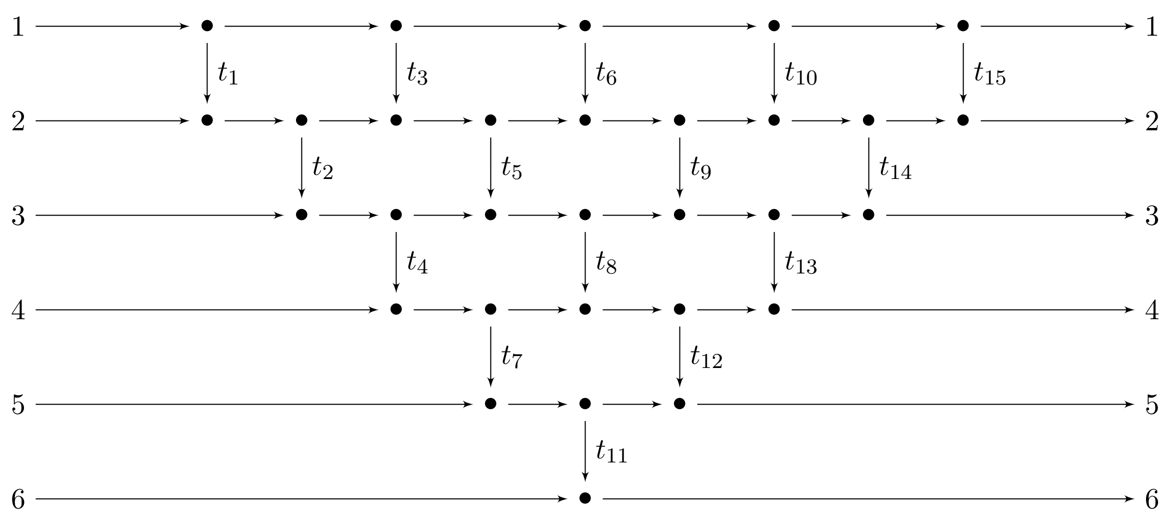
Today's goal is to give a parametrization of the set of $n \times n$ matrices of the form

$$\begin{bmatrix} 1 & * & * & \cdots & * & * \\ & 1 & * & \cdots & * & * \\ & & 1 & \cdots & * & * \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & * \\ & & & & & 1 \end{bmatrix}$$

for which all minors are positive, except the ones that are forced to be zero.

Warm up question: Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Give a simple criterion for when $\Delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k}$ can be nonzero on an upper-triangular matrix.

We now come to our main result. Let G_n be the directed graph given below by example in the case $n = 6$:



Here the horizontal edges have weight 1.

Theorem: Let the weights of the above graph range over $\mathbb{R}_{>0}^{\binom{n}{2}}$. This gives a homeomorphism between $\mathbb{R}_{>0}^{\binom{n}{2}}$ and the space of $n \times n$ matrices of the form $\begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$ for which $\Delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k}(M) > 0$ whenever $i_1 \leq j_1$, $i_2 \leq j_2$, $\ldots$, $i_k \leq j_k$.