

WORKSHEET 5: THE PRODUCT LANDS IN THE RIGHT SPACE

Let M be a $k \times n$ matrix of rank k ; let M_1, M_2, \dots, M_n be the columns of M . We'll be working with $k \times k$ minors of M , so we'll abbreviate $\Delta_I^{[k]}(M)$ to $\Delta_I(M)$.

Let $1 \leq q_1 < q_2 < \dots < q_k \leq n$ be the unique indices such that M_{q_j} is not in the span of $M_{q_j+1}, M_{q_j+2}, \dots, M_n$. (Note that the span of the empty set is $\{0\}$.) We'll call $\{q_1, \dots, q_k\}$ the **right hand pivot set** of M .

Problem 5.1. Show that $\Delta_{q_1 q_2 \dots q_k}(M) \neq 0$. Show that, if $1 \leq r_1 < r_2 < \dots < r_k \leq n$ is a set of indices with $\Delta_{r_1 r_2 \dots r_k}(M) \neq 0$, then $r_a \leq q_a$ for each a .

Problem 5.2. Let $M_{a(a+1)\dots n}$ be the matrix made of the last $n - a + 1$ columns of M . Show that $\text{rank } M_{a(a+1)\dots n} = \#(\{a, a + 1, \dots, n\} \cap Q)$.

Recall that $x_i(t)$ is $Id_n + te_{i(i+1)}$, where $e_{i(i+1)}$ has a 1 in position $(i, i + 1)$ and 0's everywhere else.

Problem 5.3. Let M be a totally nonnegative $k \times n$ matrix of rank k with right hand pivot set Q . Let $t > 0$. Show that the right hand pivot set of $Mx_i(t)$ is

$$\begin{array}{ll} Q \cup \{i + 1\} \setminus \{i\} & \text{if } i \in Q, i + 1 \notin Q. \\ Q & \text{otherwise} \end{array}$$

Recall from the homework the action of the 0-Hecke monoid on k -element subsets of $[n]$ by

$$Qe_i = \begin{cases} Q \cup \{i + 1\} \setminus \{i\} & i \in Q, i + 1 \notin Q. \\ Q & \text{otherwise} \end{cases}.$$

Let i_1, i_2, \dots, i_N be a word in the letters $\{1, 2, \dots, n - 1\}$ and let $t_1, t_2, \dots, t_N \in \mathbb{R}_{>0}$.

Problem 5.4. Let M be the top k -rows of

$$x_{i_1}(t_1) \cdots x_{i_N}(t_N).$$

Show that the right hand pivot set of M is $[k]e_{i_1}e_{i_2} \cdots e_{i_N}$.

Problem 5.5. Let the element $e_{i_1}e_{i_2} \cdots e_{i_N}$ of the 0-Hecke monoid correspond to $w \in S_n$. Show that the upper right submatrices of

$$x_{i_1}(t_1) \cdots x_{i_N}(t_N)$$

have the subranks as those of M .

Finally, we explain why we call Q the right hand pivot set. Let's define a matrix to be in **rightwards reduced row echelon form** if it looks like the following:

$$\begin{bmatrix} * & * & 0 & * & * & 0 & * & 1 & 0 \\ * & * & 0 & * & * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns with 1's in them are called the pivot columns.

For every $k \times n$ matrix M of rank k , there is a unique invertible $k \times k$ matrix g such that gM is in rightwards reduced row echelon form. The pivot positions of this form will be the righthand pivot set of M .

Of course, one can work out analogous statements using leftwards reduced row echelon form (which is the standard one in linear algebra textbooks), upwards reduced column echelon form or downwards reduced column echelon form. These are related to $My_i(t)$, $x_i(t)M$ and $y_i(t)M$, where $y_i(t) = Id_n + te_{(i+1)i}$. Sometimes, you will need a variant 0-Hecke action where $e_iQ = Q \cup \{i\} \setminus \{i + 1\}$ instead of $Q \cup \{i + 1\} \setminus \{i\}$. It's a good exercise to work out these variants.