WORKSHEET 6: BRUHAT DECOMPOSITION

We define the following subgroups of GL_n:

$$B_+ = \begin{bmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix} \qquad B_- = \begin{bmatrix} * & & & & \\ * & * & & & \\ * & * & * & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & * & * & * & * \end{bmatrix} .$$

$$N_+ = \begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & 1 & \cdots & * \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \qquad N_- = \begin{bmatrix} 1 & & & & \\ * & 1 & & & \\ * & * & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & * & * & * & 1 \end{bmatrix} .$$

We'll write $B_+(n)$ etcetera when the size of the matrix is not clear from context.

Last time we stated the Borel decomposition of GL_n:

$$\mathrm{GL}_n = \bigsqcup_{w \in S_n} B_- w B_+ .$$

And we stated that, more explicitly: $B_- w B_+$ is the set of matrices where the ranks of all upper left submatrices match the rank of w.

We will want to prove this and, for inductive purposes, it will be better to prove something stronger. Define a ***partial permutation matrix*** to be a $(0,1)$-matrix π where each row and column contains at most one 1. For example, here is a complete list of all 2×2 partial permutation matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

We'll write PP_{mn} for the set of $m \times n$ partial permutation matrices.

Problem 6.1. Show that the ranks of upper left submatrices are unchanged by left multiplication by B_- and right multiplication by B_+.

Problem 6.2. Let X be an $m \times n$ matrix. Show that there is a unique $m \times n$ partial permutation matrix π such that the ranks of each upper left submatrix match those of π.

Problem 6.3. Let X be an $m \times n$ matrix and let π be the partial permutation matrix whose upper left ranks match those of X. Show that there are matrices $b_- \in B_-(m)$ and $b_+ \in B_+(n)$ such that $X = b_- \pi b_+$.