WORKSHEET 7: MORE ON BRUHAT DECOMPOSITION

The representation of X as b_-wb_+ is generally not unique. We now discuss getting a unique representation. For simplicity, we switch back to the case of permutation matrices.

Here is where we are going:

Theorem. Each matrix in B_-wB_+ has a unique factorization in the forms

$$B_-w(w^{-1}N_+w \cap N_+) = (N_- \cap wN_-w^{-1})(B_-w \cap wB_+)(w^{-1}N_+w \cap N_+) = (N_- \cap wN_-w^{-1})wB_+.$$

Let $\Phi_+ = \{(i,j) : 1 \le i < j \le n\}$. For any $X \subseteq \Phi_+$, let $N_+(X)$ be $\{g \in N_+ : g_{ij} = 0 \text{ for } (i,j) \notin X\}$.

Problem 7.1. Show that $N_+ \cap wN_+w^{-1}$ is $N_+(X)$ for a certain set X, and describe X explicitly. Show that $\#(X) = \binom{n}{2} - \ell(w)$.

Problem 7.2. For any subset X of Φ_+, show that every element of N_+ has a unique factorization of the form $N_+(X)N_+(\Phi_+ \setminus X)$.

Problem 7.3. Show that every element of N_+ has a unique factorization in the form

$$(N_+ \cap w^{-1}N_-w)(N_+ \cap w^{-1}N_+w).$$

Problem 7.4. Show that every element of B_+ has a unique factorization in the form

$$(B_+ \cap w^{-1}B_-w)(N_+ \cap w^{-1}N_+w).$$

Problem 7.5. Show that, if any of the unique factorization claims in the theorem is true, then they all are true.

Problem 7.6. Show that every matrix in B_-wB_+ has at least one factorization as in the theorem.

Problem 7.7. Show that every matrix in B_-wB_+ has at most one factorization as in the theorem. Hint: You'll want to work with one of the forms with two factors.

One might want variants of this theorem for $B_{\pm_1}wB_{\pm_2}$ for any of the four sign choices. Here is the correct statement:

Theorem. Let $\pm_1$ and $\pm_2$ be two choices of $+$ or $-$. Then every matrix in $B_{\pm_1}wB_{\pm_2}$ has a unique factorization in any of the forms

$$B_{\pm_1}w(w^{-1}N_{\mp_1}w \cap N_{\pm_2}) = (N_{\pm_1} \cap wN_{\mp_2}w^{-1})(B_{\pm_1}w \cap wB_{\pm_2})(w^{-1}N_{\mp_1}w \cap N_{\pm_2}) = (N_{\pm_1} \cap wN_{\mp_2}w^{-1})wB_{\pm_2}.$$

The intersections $N_{\pm_1} \cap wN_{\mp_2}w^{-1}$ and $w^{-1}N_{\mp_1}w \cap N_{\pm_2}$ are isomorphic as manifolds to $\mathbb{R}^{\ell(w)}$ if $\pm_1 = \pm_2$ and to $\mathbb{R}^{\binom{n}{2}-\ell(w)}$ if $\pm_1 = \mp_2$.

Remark. The intersection $N_+ \cap wN_+w^{-1}$ is a Lie group. The corresponding Lie algebra is $\mathfrak{n}_+ \cap w\mathfrak{n}_+w^{-1}$, where $\mathfrak{n}_+$ is the upper triangular matrices with zeroes on the diagonal. The exponential map $\mathfrak{n}_+ \cap w\mathfrak{n}_+w^{-1} \longrightarrow N_+ \cap wN_+w^{-1}$ is an isomorphism. This is the conceptually right reason that $N_+ \cap wN_+w^{-1} \cong \mathbb{R}^{\binom{n}{2}-\ell(w)}$.

Remark. The factorization $N_+ = N_+(X)N_+(\Phi_+ \setminus X)$ is correct for any subset X of Φ_+, but $N_+(X)$ and $N_+(\Phi_+ \setminus X)$ are not always groups. In fact, the sets X for which $N_+(X)$ and $N_+(\Phi_+ \setminus X)$ are both groups are precisely those arising from permutations w.