WORKSHEET 9: INVERTING THE UNIPOTENT PRODUCT I:

We want to understand products

$$x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_N}(t_N)$$

where $x_j(t)$ is the $n \times n$ matrix with t in position $(j, j+1)$, with 1's on the diagonal, and 0's everywhere else. Let g_j be the partial product

$$g_j = x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_j}(t_j).$$

We are eventually (guided by our earlier work) going to work with right justified minors, which are invariant for the right action of N_-. So it is natural to think about cosets for GL_n/B_-, which we can think of as flags.

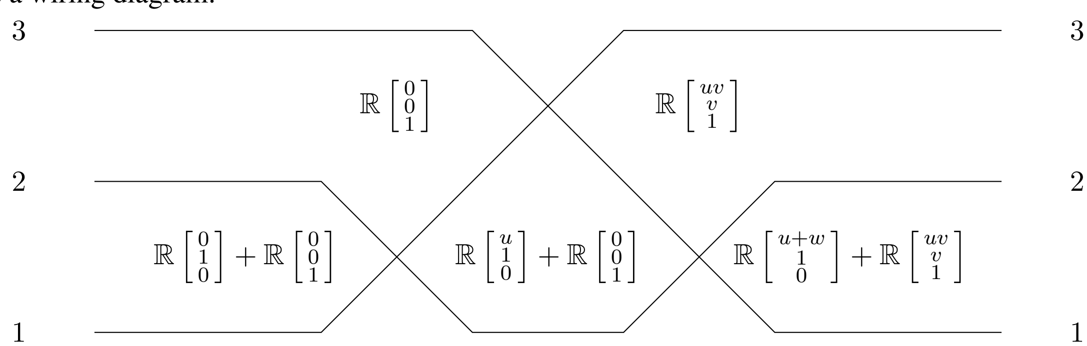
Let $(F^1_j, F^2_j, \ldots, F^{n-1}_j)$ be the flag where F^k_j is spanned by the rightmost k columns of g_j.

Problem 9.1. Show that $F^k_{j-1} = F^k_j$ for $k \neq n - i_j$. (This is mostly a matter of unwinding definitions. Also, apologies for the $n - i_j$. If I has used unipotent products in N_- on day one, it would have been i_j.)

We can visualize this by labeling each chamber of a wiring diagram with the corresponding vector space. For example, here are the partial products of $x_1(u)x_2(v)x_3(w)$

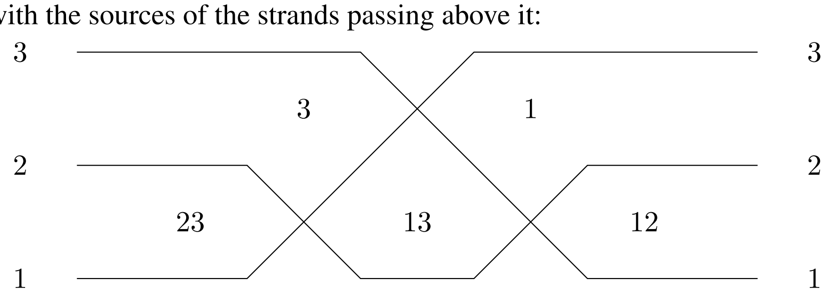
$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & u & uv \\ & 1 & v \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & u+w & uv \\ & 1 & v \\ & & 1 \end{bmatrix}$$

and here is a wiring diagram:



At each point in the partial product, the corresponding flag can be seen by reading down the corresponding chambers.

Label each chamber with the sources of the strands passing above it:



Problem 9.2. Let V be the label of a chamber at height k. Show that $\Delta^{(k+1)(k+2)\cdots n}(V) \neq 0$. (This should be easy.)

Problem 9.3. Suppose that the word $s_{i_1}s_{i_2}\cdots s_{i_N}$ is reduced. Let V be the subspace in a chamber, and let I be the set of strands labeling the chamber. Show that I is the topward pivots of V. (This is putting together a lot of earlier things.)

Problem 9.4. Label each of the chambers in the above diagram with the ratio

$$\frac{\Delta^I(V)}{\Delta^{(k+1)(k+2)\cdots n}(V)}.$$