PROBLEM SET ONE: DUE FRIDAY, SEPTEMBER 11

Problem 1. This problem is meant to give you some practice using the Lindström-Gessel-Vienott lemma.

(1) Consider the $n \times n$ matrix $X_{ij} = \binom{i+j}{i}$ for $0 \leq i, j \leq n-1$. Show that X is totally nonnegative.
(2) Consider the $n \times n$ matrix $Y_{ij} = \binom{i}{j}$ for $0 \leq i, j \leq n-1$. Show that Y is totally nonnegative.

Problem 2. Let X be a $p \times q$ matrix and Y a $q \times r$ matrix. Show that, if X and Y are totally nonnegative, then XY is a totally nonnegative.

Problem 3. This next problem features some lemmas about the symmetric group which we will want soon. Given a permutation σ of $\{1, 2, \dots, n\}$, we set $\ell(\sigma) = \#\{(i, j) \ : \ 1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j)\}$. For $1 \leq i \leq n-1$, let s_i be the permutation with $s_i(i) = i+1$, $s_i(i+1) = i$ and $s_i(j) = j$ for $j \neq i, i+1$.

(1) Let σ be any permutation and let $1 \leq i \leq n-1$. Show that $\ell(s_i \sigma) = \ell(\sigma) \pm 1$ and $\ell(\sigma s_i) = \ell(\sigma) \pm 1$ for some choice of signs. (The two signs don't have to be the same.
(2) Let σ be a permutation other than the identity. Show that there is some i with $\ell(\sigma s_i) = \ell(\sigma) - 1$ and that there is some j with $\ell(s_j \sigma) = \ell(\sigma) - 1$.
(3) Let σ be a permutation. A ***word*** for σ is a sequence $j_1, j_2, \dots, j_r$ such that $\sigma = s_{j_1} s_{j_2} \cdots s_{j_r}$; the length of this word is r. Show that there is a word for σ with length $\ell(\sigma)$, and that there are no words with any shorter length.

Problem 4. This next problem features some lemmas about ranks of submatrices which we will want soon. Let X be a $n \times n$ matrix. For $0 \leq a, b \leq n$, let $r_{ab}(X)$ be the rank of the $a \times b$ submatrix in the upper left of X. (If a or b is 0, then $r_{ab} = 0$.)

(1) Show that $0 \leq r_{(a+1)b}(X) - r_{ab}(X) \leq 1$ and $0 \leq r_{a(b+1)}(X) - r_{ab}(X) \leq 1$.
(2) Show that we cannot have $r_{ab}(X) + 1 = r_{(a+1)b}(X) = r_{a(b+1)}(X) = r_{(a+1)(b+1)}(X)$.
(3) Show that there are precisely $n!$ matrices r_{ab} with $r_{k0} = r_{0k} = 0$, $r_{kn} = r_{nk} = k$ and obeying the conditions of parts (1) and (2).